Linearization

Last class we began to apply what we know about linear systems to nonlinear systems.

**Example.** Consider the van der Pol equation

\[
\frac{d^2 x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0.
\]

The corresponding system is

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= (1 - x^2)y - x.
\end{align*}
\]

We calculated the equilibria and determined that the only equilibrium point is \((0, 0)\), and the linearized system near \((0, 0)\) is

\[
\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} Y.
\]

**Example.** Consider the (undamped) pendulum

\[
\frac{d^2 \theta}{dt^2} + \sin \theta = 0.
\]

The corresponding system is

\[
\begin{align*}
\frac{d\theta}{dt} &= v \\
\frac{dv}{dt} &= -\sin \theta.
\end{align*}
\]

There are equilibria at \((\theta, v) = (k\pi, 0)\) for all integers \(k\).

The linearized system near \((0, 0)\) is

\[
\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y,
\]

and the linearized system near \((\pi, 0)\) is

\[
\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y,
\]
Given the (nonlinear) system
\[
\frac{dx}{dt} = f(x, y) \\
\frac{dy}{dt} = g(x, y),
\]
its **Jacobian** at the point \((x_0, y_0)\) is the matrix
\[
J(x_0, y_0) = \begin{pmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
\]
and its linearization at \((x_0, y_0)\) is the system
\[
\frac{dY}{dt} = JY.
\]

For the pendulum, we have one linearization for each equilibrium point:
For the van der Pol equation, we obtain the linearization:

**Linearization Theorem**  Let $Y_0$ be an equilibrium point for the nonlinear autonomous system

$$\frac{dY}{dt} = F(Y)$$

and let

$$\frac{dY}{dt} = JY$$

be the corresponding linearized system. If the eigenvalues of $J$ are not purely imaginary, then the solution curves of the nonlinear system near $Y_0$ behave in the same qualitative way as the solution curves of the linear system.

**Example.** Consider the van der Pol equation near the origin. The linearized system is

$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} Y.$$
**Example.** Consider the pendulum equation. The linearized system near \((\pi, 0)\) is

\[
\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y.
\]

The linearized system near \((0, 0)\) is

\[
\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y.
\]
What is special about the case of purely imaginary eigenvalues in the linearization?

**Example.** Consider the one-parameter family of systems

\[
\frac{dx}{dt} = -y + \alpha x (x^2 + y^2) \\
\frac{dy}{dt} = x + \alpha y (x^2 + y^2)
\]

where \( \alpha \) is a parameter. Note that \((0,0)\) is always an equilibrium point.