The Laplace transform

For the remainder of the semester, we are going to take a somewhat different approach to the solution of differential equations. We are going to study a way of transforming differential equations into algebraic equations.

We begin with a little review of improper integrals.

**Example.** Consider the improper integral

\[ \int_0^\infty e^{-2t} \, dt. \]
**Example.** Consider the improper integrals

\[ \int_0^\infty e^{-st} \, dt \]

for various values of \( s \).

**Definition.** The *Laplace transform* of the function \( y(t) \) is the function

\[ Y(s) = \int_0^\infty y(t) e^{-st} \, dt. \]

This transform is an “operator” (a function on functions). It transforms the function \( y(t) \) into the function \( Y(s) \).
Notation: We often represent this operator using the script letter $\mathcal{L}$. In other words,

$$\mathcal{L}[y] = Y.$$  

For example, $\mathcal{L}[1] = \frac{1}{s}$.

Note that, even if $y(t)$ is defined for all $t$, the Laplace transform $Y(s)$ may not be defined for all $s$.

**Example.** Let’s compute $\mathcal{L}[e^{at}]$ using the definition and the improper integrals we have already computed:

**Examples.** Using *Mathematica* to calculate the improper integrals, we see that:

$$\mathcal{L}[\sin t] = \frac{1}{s^2 + 1} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[e^{2t} \sin 3t] = \frac{3}{s^2 - 4s + 13} \quad \text{for} \quad s > 2$$

$$\mathcal{L}[t^4] = \frac{24}{s^5} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 4} \quad \text{for} \quad s > 0,$$

$$\mathcal{L}[t \cos \sqrt{2} t] = \frac{s^2 - 2}{(s^2 + 2)^2} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[e^{\omega t}] = \frac{1}{s - i\omega} \quad \text{for} \quad s > 0$$
Properties of the Laplace transform  There are two properties of the Laplace transform that make it well suited for solving linear differential equations:

1. $\mathcal{L} \left[ \frac{dy}{dt} \right] = s\mathcal{L}[y] - y(0)$

2. $\mathcal{L}$ is a linear transform

Both of these properties are extremely important, but the surprising one is #1. Let’s consider

$$\mathcal{L} \left[ \frac{dy}{dt} \right] = \int_0^\infty \left( \frac{dy}{dt} \right) e^{-st} dt.$$  

In fact, before we consider the improper integral, let’s apply the method of integration by parts to the indefinite integral

$$\int \left( \frac{dy}{dt} \right) e^{-st} dt.$$
Now let’s see how we can use the Laplace transform to solve an initial-value problem.

**Example.** Solve the IVP

\[
\frac{dy}{dt} - 3y = e^{2t}, \quad y(0) = 4.
\]

1. Transform both sides of the equation:

2. Solve for \( \mathcal{L}[y] \):

3. Calculate the inverse Laplace transform:

Is this the right answer? Do we need Laplace transforms to calculate it?