Laplace transforms and second-order linear equations

So far we have only applied the Laplace transform to first-order equations. Now we consider second-order equations.

Recall the rule for Laplace transforms of derivatives: $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$. What does this rule say about $\mathcal{L}\left[\frac{d^2y}{dt^2}\right]$?

Now that we have this rule, we also need to add to our table of Laplace transforms. Since sine and cosine often appear as parts of the solutions to second-order equations, let's determine their Laplace transforms.

There are a number of ways to compute these transforms—using integration by parts, using Euler's formula, and even using the fact that sine and cosine are solutions to certain very special second-order equations. *Mathematica* tells us that

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega}.$$

Let's use this fact to determine $\mathcal{L}[\sin \omega t]$ and $\mathcal{L}[\cos \omega t]$.

Now that we know the transforms of sine and cosine, let's see how we use them.

Example. Compute
$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+9}\right].$$

Now for a little practice with the third rule for transforms:

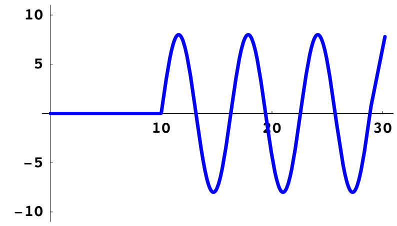
Example. Compute

$$\mathcal{L}^{-1}\left[\frac{8e^{-10s}}{(s^2+9)(s^2+1)}\right].$$

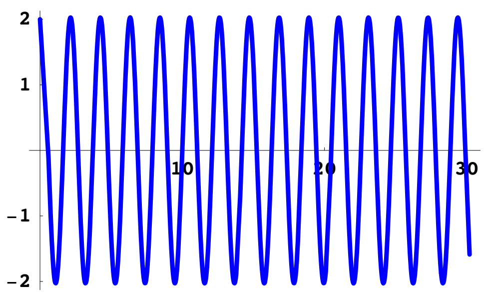
Now let's use what we have learned to solve the initial-value problem

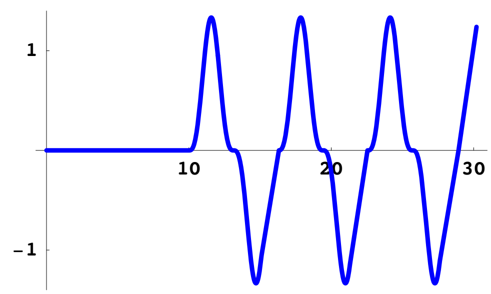
$$\frac{d^2y}{dt^2} + 9y = 8u_{10}(t)\sin(t-10), \quad y(0) = 2, \quad y'(0) = 1.$$

Here is the graph of the forcing function $8u_{10}(t)\sin(t-10)$:

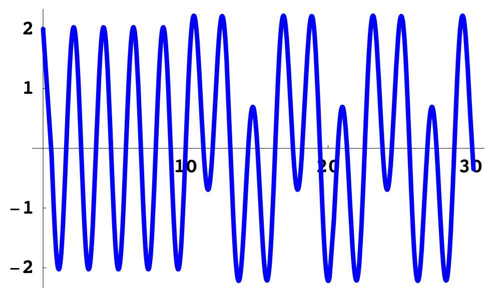


Here are the graphs of the two functions that combine to give us the desired solution.





Here is the graph of the solution



The second-order equation that we just considered is undamped. In order to consider the full range of second-order equations, we need one more property of the transform.

Shifting the s-axis. Let $Y(s)$ denote the Laplace transform $\mathcal{L}[y(t)]$. Then

$$\mathcal{L}[e^{at}\, y(t)] =$$

Example 1. Calculate $\mathcal{L}[e^{-2t}\cos 3t]$.

Example 2. Calculate $\mathcal{L}^{-1}\left[\dfrac{2s+7}{s^2+4s+7}\right]$.