Laplace transforms and second-order linear equations

So far we have only applied the Laplace transform to first-order equations. Now we consider second-order equations.

Recall the rule for Laplace transforms of derivatives: \( \mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) \). What does this rule say about \( \mathcal{L}\left[\frac{d^2y}{dt^2}\right] \)?

Now that we have this rule, we also need to add to our table of Laplace transforms. Since sine and cosine often appear as parts of the solutions to second-order equations, let’s determine their Laplace transforms.

There are a number of ways to compute these transforms—using integration by parts, using Euler’s formula, and even using the fact that sine and cosine are solutions to certain very special second-order equations. Mathematica tells us that

\[
\mathcal{L}[e^{\omega t}] = \frac{1}{s - \omega i}.
\]

Let’s use this fact to determine \( \mathcal{L}[\sin \omega t] \) and \( \mathcal{L}[\cos \omega t] \).
Now that we know the transforms of sine and cosine, let’s see how we use them.

**Example.** Compute

\[ \mathcal{L}^{-1} \left[ \frac{2s + 1}{s^2 + 9} \right]. \]
Now for a little practice with the third rule for transforms:

**Example.** Compute

\[
\mathcal{L}^{-1} \left[ \frac{8e^{-10s}}{(s^2 + 9)(s^2 + 1)} \right].
\]
Now let’s use what we have learned to solve the initial-value problem

\[
\frac{d^2 y}{dt^2} + 9y = 8u_{10}(t) \sin(t - 10), \quad y(0) = 2, \quad y'(0) = 1.
\]

Here is the graph of the forcing function $8u_{10}(t) \sin(t - 10)$:
Here are the graphs of the two functions that combine to give us the desired solution.

Here is the graph of the solution

The second-order equation that we just considered is undamped. In order to consider the full range of second-order equations, we need one more property of the transform.

**Shifting the s-axis.** Let $Y(s)$ denote the Laplace transform $\mathcal{L}[y(t)]$. Then

$$\mathcal{L}[e^{at}y(t)] =$$
Example 1. Calculate $\mathcal{L}[e^{-2t}\cos 3t]$. 

Example 2. Calculate $\mathcal{L}^{-1}\left[\frac{2s + 7}{s^2 + 4s + 7}\right]$. 