More on using the Laplace transform to solve certain second-order equations

The second-order equations that we have considered so far with the Laplace transform have been undamped. In order to consider the full range of second-order equations, we need one more property of the transform.

**Shifting the s-axis.** Let $Y(s)$ denote the Laplace transform $\mathcal{L}[y(t)]$. Then

$$\mathcal{L}[e^{at}y(t)] =$$

**Example 1.** Calculate $\mathcal{L}[e^{-2t}\cos 3t]$.

**Example 2.** Calculate $\mathcal{L}^{-1} \left[ \frac{2s + 7}{s^2 + 4s + 7} \right]$. 
Let’s solve the initial-value problem

\[ \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 7y = 10 u_3(t) \sin 2(t - 3), \quad y(0) = 2, \quad y'(0) = -1. \]

Before we get too far into the messy formulas, let’s look at the graph of the solution using HPGSystemSolver:

![Graph of the solution](image)

Now for the formulas:

1. Transform both sides of the equation:

2. Solve for \( \mathcal{L}[y] \):
3. Calculate the inverse Laplace transform:

We calculated

\[
\mathcal{L}^{-1} \left[ \frac{2s + 7}{s^2 + 4s + 7} \right] = 2e^{-2t} \cos \sqrt{3}t + \sqrt{3} e^{-2t} \sin \sqrt{3}t
\]

in Example 2.

To invert the second term, we take advantage of some algebra done before class:

(a) Partial fractions decomposition:

\[
\frac{1}{(s^2 + 4)(s^2 + 4s + 7)} = \frac{1}{73} \left( \frac{4s + 13}{s^2 + 4s + 7} - \frac{4s - 3}{s^2 + 4} \right)
\]

(b) Inverse related to the first term:

\[
\mathcal{L}^{-1} \left[ \frac{4s + 13}{s^2 + 4s + 7} \right] = 4e^{-2t} \cos \sqrt{3}t + \frac{5\sqrt{3}}{3} e^{-2t} \sin \sqrt{3}t
\]

(c) Inverse related to the second term:

\[
\mathcal{L}^{-1} \left[ \frac{4s - 3}{s^2 + 4} \right] = 4 \cos 2t - \frac{3}{2} \sin 2t
\]

After we put all of this together, we get the solution

\[
y(t) = 2e^{-2t} \cos \sqrt{3}t + \sqrt{3} e^{-2t} \sin \sqrt{3}t + \\
\frac{20}{73} u_3(t) \left( 4e^{-2(t-3)} \cos \sqrt{3}(t-3) + \frac{5\sqrt{3}}{3} e^{-2(t-3)} \sin \sqrt{3}(t-3) - 4 \cos 2(t-3) + \frac{3}{2} \sin 2(t-3) \right)
\]