Quantum trajectories from experiments to recent results

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Single measurement: For a state $\psi \in L^2(\mathbb{R}^d)$ what is the probability of finding the particle in a box E?

$$\mathbb{P}(x \in E) = \int_E |\psi(x)|^2 \underset{\cdot}{x}.$$

Repeated Measurement: Consider a protocol:

1. Measure the position (record x)
2. Evolve freely by $H = p^2$ for time t
3. Measure the position again (record x_t)

What is the probability $\mathbb{P}(x \in E, x_t \in F)$?

There is no (unique) answer to the question. We need a formalism that conveniently encodes all the answers.

Kraus Operator Formalism

Objects:

- Set Ω of possible measurement results ζ.
- Prior measure μ (and sigma algebra $\mathcal{F}$) on Ω.
- Hilbert space $\mathcal{H}$ of the system.

Definition (Kraus operators V_ζ)

A measurable function $\zeta \in \Omega \to V_\zeta \in B(\mathcal{H})$ satisfying

$$\int_\Omega V_\zeta^* V_\zeta \, d\mu(\zeta) = 1.$$

Meaning: Given a measurement result ζ the state jumps as $\psi \to V_\zeta \psi$:

1. $\mathbb{P}(\zeta \in E) = \|V_\zeta \psi\|^2 = (\psi, V_\zeta^* V_\zeta \psi)$,
2. The normalized state after the measurement is $\frac{V_\zeta \psi}{\|V_\zeta \psi\|}$.

Example (Projection Measurements)

$\Omega = \{1, \dots k\}$, $V_j = P_j$, *projections* P_j *form an orthogonal decomposition of identity,* $P_1 + \cdots + P_k = 1$.

Example (Unitary Evolution)

$\Omega = \{1\}$ *and* V_1 *is unitary.*

Measurement of Position II

Example (A measurement of position)

$\Omega = \mathbb{R}$, $V_x = \frac{1}{(\sigma\sqrt{2\pi})^{\frac{d}{2}}} \exp(-\frac{1}{2}\frac{(Q-x)^2}{2\sigma^2})$, *$Q$ is the position operator. For the average measurement result and the variance we get:*

$$\int_{\mathbb{R}^d} x\|V_x\psi\|^2 dx = (\psi, Q\psi), \quad \int_{\mathbb{R}} x^2\|V_x\psi\|^2 dx = (\psi, Q^2\psi) + \sigma^2.$$

Example (General Measurement of Position)

$\Omega = \mathbb{R}^{\mathrm{d}}$, $V_x = \sqrt{p(Q-x)}$ *where p is a probability distribution.*

What is the probability $\mathbb{P}(x \in E, x_t \in F)$?

$$\mathbb{P}(x \in E, x_t \in F) = \int_{x\in E, x_t \in F} \|V_{x_t} e^{-itp^2} V_x\psi\|^2 dx dx_t.$$

Quantum Trajectories

Given Kraus operators V_ζ, the map

$$(\zeta_1, \ldots, \zeta_n) \to V_{\zeta_n} \ldots V_{\zeta_1} \psi$$

is the quantum trajectory.

Basic Objects:

1. $\mathbb{P}_\psi(\zeta_1 \in E_1, \ldots, \zeta_n \in E_n) = \int_{E_1 \times \cdots \times E_n} \|V_{\zeta_n} \ldots V_{\zeta_1} \psi\|^2$ defines a probability measure on $\Omega^{\mathbb{N}}$.
2. Process $\psi_n := \frac{V_{\zeta_n} \ldots V_{\zeta_1} \psi}{\|V_{\zeta_n} \ldots V_{\zeta_1} \psi\|}$ is a Markov process on $(\Omega^{\mathbb{N}}, \mathbb{P}_\psi)$.

Quantum trajectories theory studies limit properties of $\mathbb{P}_\psi$ and ψ_n.

A bit of history and an example

Quantum trajetories defined an derived from the end of the 70's to the 90's by Davies, Kraus, Barchielli, Lanz, Prosperi, Lupieri, Holevo, Belavkin on the mathematical physics side.

Physics pioneer are Gisin and Diosi, and Dalibard, Castin and Molmer in the eighties and nineties. They used trajectories respectively as model for wave function collapse or as numerical tools.

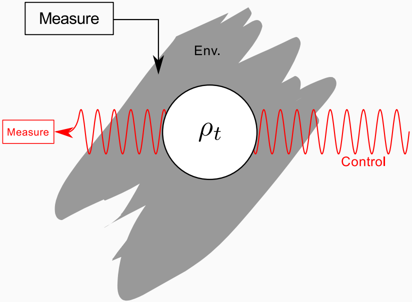
A first basic example: Radioactive decay.
$\rho_t \in M_2(\mathbb{C})$ with P_e the excited state and P_g the ground state:

$$d\rho_t = \lambda(\sigma_-\rho_{t-}\sigma_+ - \frac{1}{2}\{P_e, \rho_{t-}\})dt + \left(\frac{\sigma_-\rho_{t-}\sigma_+}{\mathrm{tr}(P_e\rho_{t-})} - \rho_{t-}\right)[dN_t - \lambda\,\mathrm{tr}(P_e\rho_{t-})dt]$$

with $\rho_0 = P_e$.

$$\text{Then prob.}(\rho_t = P_g) = 1 - e^{-\lambda t}.$$

Continuous time quantum trajectories



Evolution:

$$\begin{aligned} d\rho_t =& L(\rho_{t-})dt \\ &+ (D(\rho_{t-}) - \mathrm{tr}(D(\rho_{t-}))\rho_{t-})dB_t \\ &+ (J(\rho_{t-}) - \rho_{t-})(dN_t - \mathrm{tr}(C^*C\rho_{t-})dt). \end{aligned}$$

Signal:

Currents: $dY_t = dB_t + \mathrm{tr}(D(\rho_{t-}))dt$

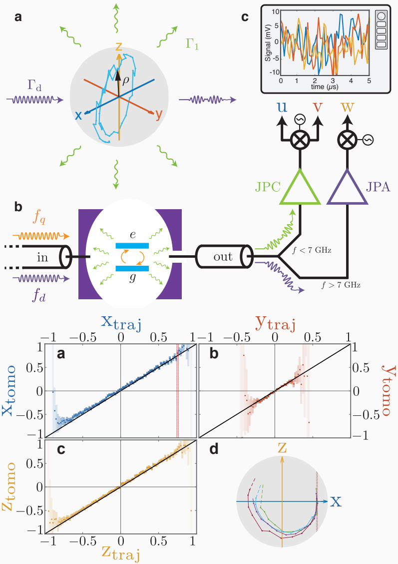
Particle counting: dN_t.

(ρ_t) is a **Markov process**.

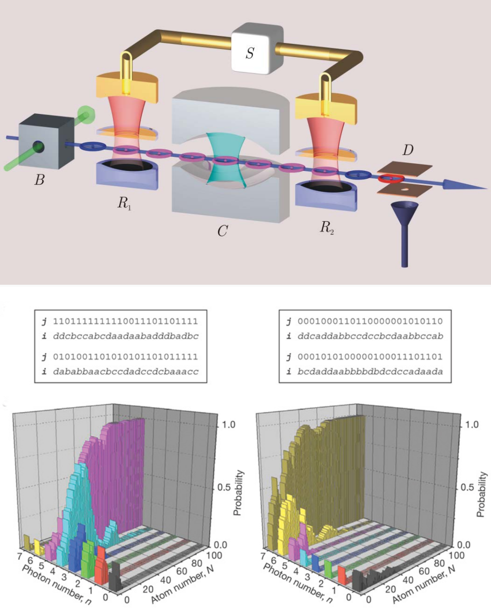
"[...] It is fair to state that we are not experimenting with single particles, any more than we can raise Ichthyosauria in the zoo. We are scrutinising records of events long after they have happened."

– E. Schrödinger, 1952.

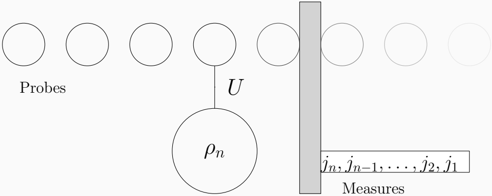
Quantum trajectories experiments: B. Huard's group (ENS Lyon)



Quantum trajectories experiments: S. Haroche's group (LKB)



Discrete time quantum trajectories



State: density matrix ρ_n
Evolution: Matrices $\{V_j\}_{j=1}^k$, $\sum_j V_j^* V_j = \mathrm{Id}$.

Measurement results distribution: $\mathbb{P}_\rho(j_1, \ldots, j_n) = \mathrm{tr}(V_{j_n} \cdots V_{j_1} \rho_0 V_{j_1}^* \cdots V_{j_n}^*)$.

$\mathbb{P}_\rho$ is a measure over a **one-sided shift of finite type**.

System evolution:
In mean: $\overline{\rho}_n = \Phi(\overline{\rho}_{n-1})$, $\Phi(\rho) = \sum_j V_j \rho V_j^*$.
Knowing $j_1, \ldots, j_{n-1}$: $\rho_n = \frac{V_j \rho_{n-1} V_j^*}{\mathrm{tr}(V_j \rho_{n-1} V_j^*)}$ with prob. $\mathrm{tr}(V_j \rho_{n-1} V_j^*)$.

The quantum trajectory (ρ_n) is a **Markov chain**.

Measurement results: ergodicity

Theorem (Kümmerer, Maassen 2000)

If $\exists! \rho$ *such that* $\Phi(\rho) = \rho$*, then,* $\mathbb{P}_\rho$ *is ergodic with respect to the left shift. Namely, for any* $i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+\ell}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}_\rho(j_1 = i_1, \ldots, j_m = i_m;\ j_{k+1} = i_{m+1}, \ldots, j_{k+\ell} = i_{m+\ell})$$
$$= \mathbb{P}_\rho(j_1 = i_1, \ldots, j_m = i_m)\mathbb{P}_\rho(j_1 = i_{m+1}, \ldots, j_\ell = i_{m+\ell}).$$

Measurement results: Entropy production and detailed balance

[B., Cuneo, Jaksic, Pautrat, Pillet, Shirikyan '17–'19]

Let $\widehat{\mathbb{P}}_n(j_1, \ldots, j_n) = \mathbb{P}_n(\theta(j_n), \ldots, \theta(j_1))$. Then,

- $ep = \lim_n \frac{1}{n} S(\mathbb{P}_n | \widehat{\mathbb{P}}_n)$ exists;
- $\sigma_n = \frac{1}{n} \log \frac{\mathbb{P}_n(j_1, \ldots, j_n)}{\widehat{\mathbb{P}}_n(j_1, \ldots, j_n)}$ verifies a LDP and fluctuation relation;
- $ep = 0$ is essentially equivalent to Φ verifying (KMS) Quantum Detailed Balance.

Trajectory: Purification

Definition (Dark subspace)
$\mathcal{K}$ is a dark subspace of $\mathcal{H}$ if $\dim \mathcal{K} \geq 2$ and for any $j_1, \dots, j_\ell$, there exists a unitary matrix $U \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{R}$ such that

$$\forall \Psi \in \mathcal{K}, V_{j_n} V_{j_{n-1}} \cdots V_{j_1} \Psi = \lambda U \Psi.$$

Theorem (Purification (Maassen, Kümerer '06))
The family of matricies $\{V_j\}_{j=1}^k$ does not have dark subspaces iff for any ρ_0,

$$\lim_{n \to \infty} S(\rho_n) = 0$$

with S the von Neumann entropy ($S(\rho) = -\operatorname{tr}(\rho \log \rho)$).

Trajectory: Purification

Refining this result, the trajectory state can be estimated using only the measurement results:

Similar results hold for continuous time trajectories (Barchielli, Paganoni '03).

Trajectory: Linear law of large numbers

Theorem (Kümerer, Maassen '04)

$$\rho_\infty = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \rho_k$$

exists almost surely and $\Phi(\rho_\infty) = \rho_\infty$.

Similar result in continuous time. Using Poisson equation techniques, one can also derive a CLT and a LIL.

Trajectory: Uniqueness of the invariant measure

Theorem (B., F., Pautrat, Pellegrini '17)

If $\{V_j\}_{j=1}^k$ has no dark subspaces, then (ρ_n) accepts a unique invariant measure $\nu_{inv.}$ iff there exists a unique ρ_∞ such that $\Phi(\rho_\infty) = \rho_\infty$.

Moreover, there exists $C > 0$ and $\lambda < 1$ such that for any ϱ,

$$W_1(\nu_n, \nu_{inv.}) \leq C\lambda^n$$

with ν_n the law of ρ_n knowing $\rho_0 = \varrho$ and W_1 the Wasserstein distance.

The LLN for any continuous function and the CLT and LIL for any Hölder continuous function can be deduced from the proof.

Similar result in continuous time.

Trajectory: Invariant measure example

Perturbation Theory, the non-demolition case

Let $V_\zeta(\phi)$ be a family of Kraus operators.

General Question: How quantum trajectories depend on the parameter ϕ?

Some results:

- Graf et. al. 2018
- Bauer, Bernard, et. al. 2015 –
- Bernardin, Chetrite, Chhaibi, Najnudel, Pellegrini 2018
- Ballesteros, Crawford, F, Frohlich, Schubnel 2017
- Benoist et. al. 2019

Example: Perturbation of Non-demolition

For two by two matrices and $\Omega = \{1, 2\}$ put $V_{1,2}(\phi)$

$$V_1 = e^{-i\phi\sigma_x} \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{q} \end{pmatrix}, \quad V_2 = e^{-i\phi\sigma_x} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}$$

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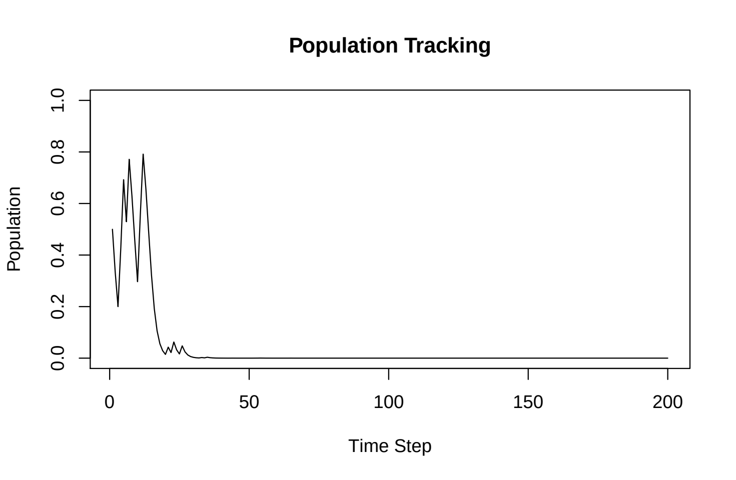
Population $\mathrm{Tr}(\rho_n \sigma_z) + 1/2$: $\phi = 0$



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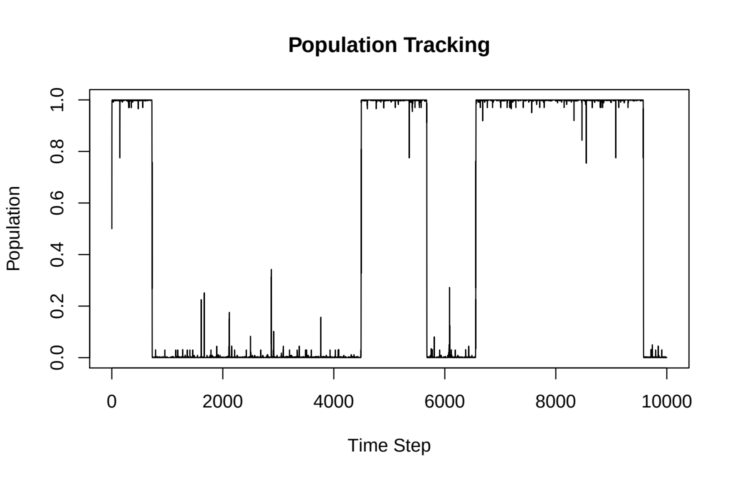
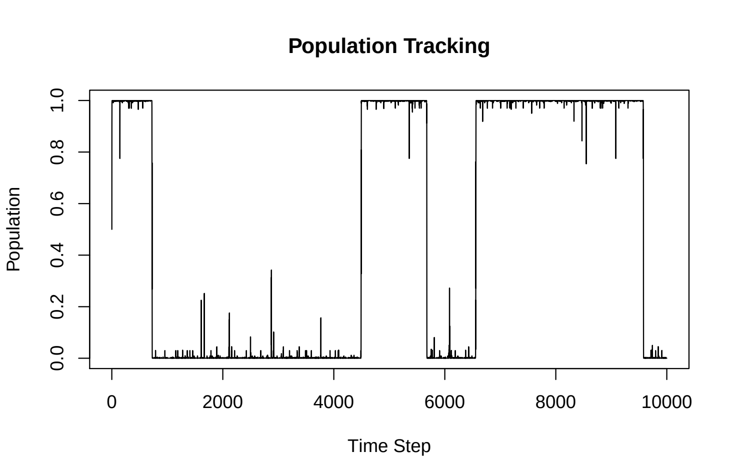
Population $\mathrm{Tr}(\rho_n\sigma_z) + 1/2$: $\phi > 0$:



Population $N_n := \mathrm{Tr}(\rho_n \sigma_z) + 1/2$ seems to approach after rescaling a jump process Y_s, $0 \leq s \leq 1$.

Negative result [Bauer, Bernard]: $N_{\phi^{-2}s}$ does not approach Y_s in Skorokhod topology as ϕ goes to zero.

Positive result [Ballesteros, Crawford, F, Fröhlich, Schubnel]: An average over $\log \phi$ time interval does.



Semiclassical Limit Problem, Intro

Infinite Dimensions: Little is known,

1. Non-demolition trajectories ($[V_\zeta, V_{\zeta'}] = 0$) [BCFFS, 2018] by mapping to a limit problem for a maximum likelihood estimator.
2. Classification of Stationary Subspaces of $K\rho = \sum_\zeta V_\zeta \rho V_\zeta^*$. [Carbone, Pautrat 2016]
3. Quantum Stochastic Diff. Eq. [Holevo's book]

Open Problems: For example,

1. Purification
2. Classification of Invariant Measures

Semiclassical Limit Problem

Describe a motion of continuously observed particle (particle detector): $\mathcal{H} = L^2(\mathbb{R}^d)$, Q is the position operator,

$$V_\zeta = e^{i\hbar t(\frac{p^2}{2M} + V(Q))}\tilde{V}_\zeta, \quad \tilde{V}_\zeta = \frac{1}{(\sigma\sqrt{2\pi})^{\frac{d}{2}}}\exp(-\frac{1}{2}\frac{(Q-\zeta)^2}{2\sigma^2}).$$

or a continuous version

$$d\psi = -\frac{i}{\hbar}H\psi dt - \frac{\gamma^2}{2}(Q - \langle Q\rangle_t)^2\psi dt + \gamma(Q - \langle Q\rangle_t)\psi dW_t,$$

with $H = -\frac{\hbar^2}{2m}\triangle + V(Q)$.

For quadratic V [Belavkin, Kolokoltsov 90's, Bassi et. al. 20's, Bauer, Bernard, Jin 2018]:

1. Construction of semiclassical wave-packets
2. Any initial state approaches a semiclassical wave-packet

Semiclassical Limit Problem

$$d\psi = -\frac{i}{\hbar} H\psi dt - \frac{\gamma^2}{2}(Q - \langle Q\rangle_t)^2 \psi dt + \gamma(Q - \langle Q\rangle_t)\psi dW_t,$$

- $H = -\frac{\hbar^2}{2m}\triangle + V(Q)$,
- Purification time scale $\tau = \gamma^{-1}\sqrt{\frac{m}{\hbar}}$,
- Purification lenght scale $l = (\frac{\hbar}{m\gamma^2})^{\frac{1}{4}}$.

Conjecture: [Bauer, Bernard, Jin 2018] Assume V smooth. In the limit $\gamma \to \infty$, $\hbar/m \to 0$ with $\epsilon = \hbar\gamma/m$ fixed, solution of the equation localizes at a position x_t that solves Langevin equations

$$dx_t = v_t dt, \quad dv_t = -\frac{1}{m}\nabla V(x_t)dt + \epsilon dW_t.$$

An open problem in perturbation theory

Let $V_\zeta(\phi)$ be a family of Kraus operators and $\mathbb{P}_\phi$ the corresponding distribution of measurement results (for the unique invariant ρ).

Question: Does Fisher information

$$\lim_{n\to\infty} \frac{F_n(\phi)}{n} := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_\phi[(\partial_\phi \log \mathbb{P}_\phi(\zeta_1, \ldots, \zeta_n))^2]$$

exists? If it exists, is it strictly positive?

Some open questions: Trajectory LDP

Does $s_n(f) = \frac{1}{n}\sum_{k=1}^{n} f(\rho_k)$ verify an LDP?

Issue with Gartner–Ellis Theorem: There is some example where the spectral radius of the twisted kernel Π_α defined by

$$\Pi_\alpha g(\rho) = \sum_j e^{\alpha f(\rho(j))} g(\rho(j))\, \mathrm{tr}(V_j \rho V_j^*) \quad \text{with } \rho(j) = \frac{V_j \rho V_j^*}{\mathrm{tr}(V_j \rho V_j^*)}$$

is not differentiable at some α for a smooth f.

Some open questions: Canonical invariant measures

Most random indirect measurement: $j \to \psi$, ψ à priori uniform.

$$\rho_{n+1} = \frac{\mathrm{tr}_{probe}[(|\psi\rangle\langle\psi| \otimes \mathrm{Id})U(|e_0\rangle\langle e_0| \otimes \rho_n)U^*(|\psi\rangle\langle\psi| \otimes \mathrm{Id})]}{\mathrm{tr}[(|\psi\rangle\langle\psi| \otimes \mathrm{Id})U(|e_0\rangle\langle e_0| \otimes \rho_n)U^*]}.$$

with

$$\mathrm{prob.}(|\psi\rangle \in A|\rho_n) = \int \mathbf{1}_{w|e_0\rangle \in A}\, \mathrm{tr}[(w|e_0\rangle\langle e_0|w^* \otimes \mathrm{Id})U(|e_0\rangle\langle e_0| \otimes \rho_n)U^*]d\mathrm{Haar}(w).$$

- Can there exist dark subspaces?
- Does this measure has a thermodynamic limit when ρ_∞ has one?
- Is the Markov chain reversible when Φ verifies (KMS) QDB?

Some open questions: Maximal entropy measures and quantum trajectories

Given a state ρ, the maximal entropy measure over pure states is given by a measure with gaussian density with respect to the uniform measure over the 1-sphere.

- Finer characterization of this measure (high temperature limit, ...)?
- Does this measure has a thermodynamic limit when Φ and ρ have one?
- Can we construct a quantum trajectory whose unique invariant measure is the maximal entropy measure?