My research interest during graduate school primarily lies in real analysis, with a particular focus on Whitney-type extension and selection problems and their applications. I am also studying the best constants for Young’s convolution inequality on nilpotent Lie groups.

**Nonnegative Interpolation and Interpolation with range restriction.** At the basic level, Whitney problems ask for efficient methods to fit a smooth function to some (random) data points while minimizing some quantities (e.g. norms, energy functionals) and obeying certain constraints (e.g. nonnegativity, convexity). The precise problem I am studying is the following.

**Problem 1.** Given finite \( E \subseteq \mathbb{R}^n \), \(-\infty \leq \lambda \leq \Lambda \leq \infty \), \( f : E \to \mathbb{R} \cap [\lambda, \Lambda] \), how do we efficiently compute the order of magnitude of

\[
\| f \|_{C^m(E; \lambda, \Lambda)} := \inf \{ \| \tilde{F} \|_{C^m(\mathbb{R}^n)} : \tilde{F} = f \text{ on } E \text{ and } \lambda \leq \tilde{F} \leq \Lambda \text{ on } \mathbb{R}^n \}
\]

and a function \( F \in C^m(\mathbb{R}^n) \) with \( F = f \text{ on } E \), \( \lambda \leq F \leq \Lambda \) on \( \mathbb{R}^n \), and \( \| F \|_{C^m(\mathbb{R}^n)} \leq C(m, n) \| f \|_{C^m(E; \lambda, \Lambda)} \)?

Solutions to Problem 1 can have abundant applications. For instance, Problem 1 addresses nonnegative interpolation if we take \( (\lambda, \Lambda) = (0, \infty) \), and it can be used to study various physical quantities such as temperature, chemical concentration, energies, etc. Problem 1 in its general formulation can be used to study trajectory design in an obstacle course.

Range-restriction problems in the broad sense have been studied in, e.g., [6,19,20,28–30]. The special case \( (\lambda, \Lambda) = (0, \infty) \) of Problem 1 was considered by Fefferman-Israel-Luli in [10,11]. The authors reduced the problem of computing the global norm (1) to computing norms on bounded subsets by means of proving a Brudnyi-Shvartsman-type Finiteness Principle [4,5], but left open the question how to compute (1) efficiently.

The Brudnyi-Shvartsman Finiteness Principle was first introduced in [4,5] to study the original Whitney extension problem (without constraint). In plain words, the Principle states that the only obstruction towards a global extension (in the sense of existence or having small norm) consists of extension problems on finite sets. See below for several compact formal statements.

In a series of three papers [21–23] with my advisor Kevin Luli, we solve Problem 1 for \( (\lambda, \Lambda) = (0, \infty) \) (i.e. nonnegative interpolation) and \( m = n = 2 \).

In [22], we obtain the following results.

- **We prove a Brudnyi-Shvartsman-type Finiteness Principle** [4,5] which can be compactly stated as follows.

\[
\| f \|_{C^2(E; 0, \infty)} \approx \max_{S \subseteq E, \# S \leq 64} \| f \|_{C^2(S; 0, \infty)}.
\]

The number 64 is a substantial improvement of the \( 5^{200} \) previously obtained by Fefferman-Israel-Luli [10,11].

- **We prove an improved Finiteness Principle**, namely,

\[
\| f \|_{C^2(E; 0, \infty)} \approx \max_{1 \leq L \leq \# E} \| f \|_{C^2(S_L; 0, \infty)},
\]

where \( L \leq \# E \), each \( \# S_L \) is universally bounded, and the \( S_L \) can be efficiently computed (using at most \( O(N \log N) \) operations and \( O(N) \) storage). See e.g., [23] and a brief discussion below.

- **We prove that in general, contrary to the unconstrained problem** [7], there does not exist a bounded additive extension operator that preserves nonnegativity. This result is first of its kind regarding the non-existence of linear extension operator.
In [21], we construct a (nonlinear) extension operator of “bounded depth”. In brief, the second-order Taylor polynomial of an interpolant at any given point can be computed from a bounded subcollection of $E$.

In [23], we provide Fefferman-Klartag-type [9,13,14] algorithmic solutions to nonnegative $C^2(\mathbb{R}^2)$ interpolation. In brief, given a finite set $E \subseteq \mathbb{R}^2$ with $\#E = N$, we can compute an interpolant using at most $O(N \log N)$ operations and $O(N)$ storage. The complexity here is likely the best possible.

Thus, we answered Problem 1 for the case $m = n = 2$ and $(\lambda, \Lambda) = (0, \infty)$.

Building on [21–23], Charles Fefferman, Kevin Luli, and I solved Problem 1 for $m = 2$ and general $-\infty \leq \lambda \leq \Lambda \leq \infty$, $n \in \mathbb{N}$ in [12]. In particular, we developed efficient algorithms for computing a $C^2(\mathbb{R}^n)$ interpolant preserving the range with the same aforementioned running time.

**Smooth convex selection.** I also study the following vector-valued variant of Problem 1.

**Problem 2.** Let $E \subseteq \mathbb{R}^n$ be an arbitrary closed set. Let $\text{Conv}(\mathbb{R}^d)$ denote the collection of convex subsets of $\mathbb{R}^d$. Let $K : E \to \text{Conv}(\mathbb{R}^d)$ be a set valued function.

(A) How do we know if there exists $F \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ with $F(x) \in K(x)$ for $x \in E$?
(B) How do we compute the order of magnitude of the smallest of such $\|F\|_{C^m(\mathbb{R}^n, \mathbb{R}^d)}$?

Fefferman-Israel-Luli solved Problem 2(B) for finite $E$ in [10] by providing a positive answer to the Brudnyi-Shvartsman conjecture (Finiteness Principle) [4,5], compactly stated as follows.

$$\inf \left\{ \|\tilde{F}\|_{C^m(\mathbb{R}^n, \mathbb{R}^d)} : \tilde{F}(x) \in K(x) \forall x \in E \right\} \leq C(m, n, d) \cdot \max_{\#S \leq k(m,n,d)} \inf \left\{ \|\tilde{F}\|_{C^m(\mathbb{R}^n, \mathbb{R}^d)} : \tilde{F}(x) \in K(x) \forall x \in S \right\}. \quad (2)$$

The techniques in [10] rely on a special concept (“shape fields”) tailored to finite sets. However, it is possible to extend the techniques in [10] to solve Problem 2 (i.e., for infinite sets $E$). This is done in my joint work in [24]. See the discussion below for more details.

Problem 2 is closely related to the following generalized version of the Brenner-Epstein-Hochster-Kollár problem.

**Problem 3.** Given $E \subseteq \mathbb{R}^n$, $\mathbb{R}$-valued functions $\phi_1, \cdots, \phi_d$, an $\mathbb{R}^s$-valued function $\phi$, and $\text{Conv}(\mathbb{R}^s)$-valued functions $K_1, \cdots, K_d$, decide if there exist $f_1, \cdots, f_d \in C^m(\mathbb{R}^n, \mathbb{R}^s)$ such that

$$\sum_{j=1}^d \phi_j(x)f_j(x) \leq \phi(x) \text{ for } x \in E \quad \text{and} \quad f_j(x) \in K_j(x) \text{ for } x \in E, \ j = 1, \cdots, d. \quad (3)$$

Pioneer works on Problem 3 with equality and without the convex constraints (i.e., $f_j(x) \in K_j(x)$) are due to Kollár [27], Fefferman-Klartár [15], Fefferman-Luli [16–18], and more recently Bierstone-Campsesato-Milman [2].

In a series of two papers [24,25], Kevin Luli, Kevin O’Neill, and I solved Problem 2. Viewing $K(x)$ as the “family of solutions at $x$” for the system (3) in Problem 3, we see that the solution to Problem 2 yields a solution to Problem 3. The main results in [24,25] also provide necessary and sufficient conditions for a real-valued function on a compact set to have a globally nonnegative $C^m$-extension.

In [24], we solved Problem 2(A) by providing a Fefferman-type criterion [8], stated in terms of “Glaeser refinement” which is a higher-dimensional generalization of the divided difference. We assign to each $x \in E$ a fiber $\Gamma(x)$ consisting of polynomial vectors $P$ such that $P(x) \in K(x)$ and $\deg P \leq m$. The Glaeser refinement of $\Gamma(x)$ removes polynomials that are not compatible with
nearby fibers (in the sense of Taylor’s theorem). We show that such bundle $(\Gamma(x))_{x \in E}$ stabilizes after a bounded number of Glaeser refinements, and we can extract a $C^m$ section from the remaining (nonempty) bundle. The main difficulties in adapting the techniques in [8] to the current setting are the loss of linearity and the presence of boundaries due to the convex constraint. To circumvent the obstacle, we localize the notion of dimension in order to establish Glaeser stability, and we replace linear analysis by convex analysis of the relative interior in order to obtain Whitney compatibility and topological compactness.

In [25], we improved the finiteness constant $k(m,n,d)$ (see (2)) in [10] to the near-optimal $k = 2^{((m+n-1) -1)}d$. Our approach is inspired by the clustering technique of Bierstone-Milman [3].

Young’s convolution inequality on nilpotent Lie groups. Beckner [1] and Klein-Russo [26], respectively, provide the best constants in Young’s convolution inequality for $\mathbb{R}^n$ and for the Heisenberg group $\mathbb{H}^n$. The former shows that extremizers are Euclidean Gaussians, while the latter shows that extremizers do not exist for the Heisenberg group or general (connected and simply-connected) nilpotent groups. In an ongoing work with Kevin O’Neill, we provide a quantitative ($o(1)$-type) characterization of near-extremizers for step-two nilpotent (Carnot) groups by showing that near extremizers are close to Gaussians in $L^p$.

Work in progress.

- Nonnegative interpolation with higher regularity.
- The structure and extension of nonnegative Sobolev functions.
- Young’s convolution inequality for step-three nilpotent groups.

References