1 Lebesgue Measure Review

The main references are Measure and Integral: An Introduction to Real Analysis by Wheeden and Zygmund, and Princeton Lectures in Analysis Vol III: Real Analysis by Stein and Shakarchi.

1.1 Motivation

Let $F$ be a family of subsets of $\mathbb{R}^d$. Our goal is to construct a size function $|\cdot| : F \to \mathbb{R} \cup \{\infty\}$ that satisfies the following reasonable properties.

1.1.1) If $K \in F$ is some "usual geometry", then $|K|$ agrees with the Euclidean volume of $K$.

1.1.2) $|E| = \sum_{n \in \mathbb{N}} |E|$ if $E = \sqcup_{n \in \mathbb{N}} E_n$. This is an upgrade from the Riemann’s finite mesh.

1.1.3) $|\cdot|$ behaves reasonably well under the symmetries of $\mathbb{R}^d$.

1.2 Outer measure

Let $R = \prod_{j=1}^d [a_j, b_j]$ be a $d$-dimensional rectangle in $\mathbb{R}^d$, we define

$$\text{vol}(R) = \prod_{j=1}^d |b_j - a_j|.$$

Let $E \subseteq \mathbb{R}^d$ be arbitrary. Let $S$ be an at most countable collection of rectangles that covers $E$. We define the (Lebesgue) exterior measure of $E$ to be

$$|E|_e := \inf \left\{ \sum_{R' \in S} \text{vol}(R') : S \text{ is an at most countable cover of } E \right\}.$$

**Proposition 1.** The following hold for exterior measure.

1.2.1) If $R$ is a rectangle, then $|R|_e = \text{vol}(R)$. This is a sanity check.

1.2.2) If $E_1 \subseteq E_2$, then $|E_1|_e \leq |E_2|_e$.

1.2.3) If $E = \bigcup_{n \in \mathbb{N}} E_n$, then $|E|_e \leq \sum_{n \in \mathbb{N}} |E_n|_e$. This is called countable sub-additivity.

**Proof.** We prove (1.2.1) and sketch the rest.

$|R|_e \leq \text{vol}(R)$ is obvious since $R$ covers itself. It suffices to show the opposite inequality.

Let $S = \{R_j : j \in \mathbb{N}\}$ cover $R$ and let $\varepsilon > 0$ be given. Let $R'_j$ be a slight enlargement of $R_j$ such that

1.2.4) $R_j \subseteq \text{interior}(R'_j)$, and

1.2.5) $\text{vol}(R'_j) \leq (1 + \varepsilon) \text{vol}(R_j)$.

Consequently, $R \subseteq \bigcup_{j \in \mathbb{N}} \text{interior}(R'_j)$. $R$ is compact by the theorem of Heine-Borel, so there exists $N \in \mathbb{N}$ such that
(1.2.6) $R \subseteq \bigcup_{j=1}^{N} \text{interior}(R_j)$.

Therefore, we have

\begin{equation}
\text{vol}(R) \leq \sum_{j=1}^{N} \text{vol}(R_j) \leq (1 + \epsilon) \sum_{j=1}^{N} \text{vol}(R_j) \leq (1 + \epsilon) \sum_{j \in \mathbb{N}} \text{vol}(R_j) = (1 + \epsilon) \sum_{R' \in \mathcal{S}} \text{vol}(R').
\end{equation}

Note that (1.2.7) holds for arbitrary countable cover $\mathcal{S}$ of $R$, we may let $\epsilon$ tend to zero and conclude the proof.

For (1.2.2), note that any cover of $E_2$ is automatically a cover of $E_1$.

For (1.2.3), use the fact that a countable union of countable sets is still countable, and use geometric series to control the error term.

**Proposition 2.** If $\text{dist}(E_1, E_2) > 0$, then $|E_1 \cup E_2|_e = |E_1|_e + |E_2|_e$.

**Proof.** $|E_1 \cup E_2|_e \leq |E_1|_e + |E_2|_e$ is obvious. For the opposite inequality, fix $\epsilon > 0$ and let $S$ cover $E_1 \cup E_2$ with $\epsilon$ error. By decomposing the rectangles that intersect both $E_1$ and $E_2$, we may assume that $S = S_1 \sqcup S_2$, where $S_i$ covers $E_i$. Then

\[ |E_1|_e + |E_2|_e \leq \sum_{R' \in S_1} \text{vol}(R') + \sum_{R'' \in S_2} \text{vol}(R'') \leq \sum_{R \in \mathcal{S}} \text{vol}(R) \leq |E_1 \cup E_2| + \epsilon. \]

Let $\epsilon$ tend to zero and we can conclude the proof.

### 1.3 Lebesgue Measure

**Definition 1.** A subset $E \subseteq \mathbb{R}^d$ is **Lebesgue measurable** if given $\epsilon > 0$, there exists an open set $G \supset E$ such that

\[ |G \setminus E|_e < \epsilon. \]

We define the **Lebesgue measure** of $E$ to be

\[ |E| = |E|_e. \]

**Proposition 3.** The following holds for Lebesgue measure and measurable sets.

(1.3.1) Every open set is measurable.

(1.3.2) Every set of exterior measure zero is measurable.

(1.3.3) Every rectangle is measurable.

(1.3.4) The collection Lebesgue measurable sets is a $\sigma$-algebra, and it contains the Borel $\sigma$-algebra.

**Sketch of proof.** The hardest part of (1.3.4) is showing that every closed set is measurable. We sketch the proof in the following.

Using the $\sigma$-compact structure of $\mathbb{R}^d$, it suffices to show that compact sets are measurable.

Let $K \subseteq \mathbb{R}^d$ be compact and let $\epsilon > 0$. Let $G \supset K$ be an open set such that
\((1.3.5) \, |G| = |G| < |K|_e + \epsilon.\)

\(G \setminus K\) is open, so we have a countable decomposition \(G \setminus K = \bigcup_{n \in \mathbb{N}} Q_n\) where \(Q_n\)'s are closed cubes, and \(\text{interior}(Q_n) \cap \text{interior}(Q_{n'}) = \emptyset\) for \(n \neq n'\). Therefore, we have

\[|G \setminus K|_e = \left| \bigcup_{j \in \mathbb{N}} Q_j \right|_e \leq \sum_{j \in \mathbb{N}} \text{vol}(Q_j).\]

It suffices to show that

\[(1.3.6) \, \sum_{j \in \mathbb{N}} \text{vol}(Q_j) \leq \epsilon.\]

Note that we have

\[(1.3.7) \, |G| \geq \left| K \cup \left( \bigcup_{j=1}^{N} Q_j \right) \right|_e = |K|_e + \sum_{j=1}^{N} \text{vol}(Q_j).\]

Let \(N\) tend to zero in (1.3.7), and (1.3.6) follows from (1.3.5) and (1.3.7).

**Proposition 4.** The collection of Lebesgue measurable sets is invariant under affine transformations.

**Proof.** Induction on rectangles. Note that finite dimensional affine transformations are automatically Lipschitz.

\[\square\]

### 2 Cantor Set

We focus on the classical one-third Cantor set. Higher dimensional analogues include Cantor dust, Sierpinski carpet, Menger sponge etc..

#### 2.1 Construction

Let \(C_0 = [0, 1]\). Construct \(C_k (k \geq 1)\) as follows.

\[(2.1.1) \, \text{Let } \{I_j\} \text{ be the collection of disjoint closed interval in } C_{k-1}. \text{ For each } j, \text{ let } I_j^{(1)} \text{ and } I_j^{(2)} \text{ be the left and right one-third closed interval of } I_j. \text{ Let } C_k = \bigcup_j \bigcup_{i \in \{1,2\}} I_j^{(i)}.\]

Define the Cantor set \(C\) by

\[C := \bigcap_{k=0}^{\infty} C_k.\]

**Proposition 5.** \(C\) is nonempty, compact, totally disconnected, and has no isolated point. In particular, it is perfect (closed and having no isolated point).
2.2 The Cantor-Lebesgue function

Proposition 6. Every number \( x \in [0, 1] \) has a (not necessarily unique) ternary expansion

\[
x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \text{ where } a_k \in \{0, 1, 2\}.
\]

\( x \in \mathcal{C} \) if and only if the representation \((2.2.1)\) is unique, and \( a_k \in \{0, 2\} \).

Proof. A point in the Cantor set is specified by a sequence of LEFT or RIGHT. \(\square\)

Definition 2. The Cantor Lebesgue function, defined on \(\mathcal{C}\), is given by

\[
\text{Can}(x) := \sum_{k=1}^{\infty} b_k \frac{2^k}{2^k} \text{ for } x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \text{ and } b_k = \frac{a_k}{2}.
\]

Proposition 7. \(\text{Can} : \mathcal{C} \to [0, 1] \) is a surjection.

Proposition 8. One can extend \(\text{Can}\) to the whole of \([0, 1]\) such that the extension is constant on each interval of \([0, 1] \setminus \mathcal{C}\).

3 A Non-measurable Set

We need a technical but useful lemma.

Lemma 1. Let \( E \subseteq \mathbb{R} \) be a measurable set with positive measure. Then the difference set \( \Delta E := \{x - y : x, y \in E\} \) contains an open interval centered at the origin.

Theorem 1 (Vitali). There exists a nonmeasurable set.

Proof. Define \(\sim\) on \([0, 1]\) by \(x \sim y\) if \(x - y \in \mathbb{Q}\). \([0, 1]/\mathbb{Q}\) admits uncountably many equivalence classes.

Invoke Axiom of Choice, and let \(\mathcal{N}\) consist of exactly one element from each distinct equivalence class.

The claim is that \(\mathcal{N}\) is not measurable.

Suppose towards a contradiction, that \(\mathcal{N}\) is measurable. Let \(\{q_k\}\) be an enumeration of \(\mathbb{Q} \cap [-1, 1]\), and let \(\mathcal{N}_k = \mathcal{N} + q_k\). The \(\mathcal{N}_k\)’s are disjoint, since distinct equivalent classes must differ by an irrational. Moreover,

\[
[0, 1] \subseteq \bigcup_{k=1}^{\infty} \mathcal{N}_k \subseteq [-1, 2].
\]

Since measurable sets are invariant under translation, we have

\[
1 \leq \sum_{k=1}^{\infty} |\mathcal{N}_k| = \sum_{k=1}^{\infty} |\mathcal{N}| \leq 3.
\]

Neither \(|\mathcal{N}| = 0 \) nor \(|\mathcal{N}| > 0\) is possible, and we have gotten ourselves into a mess. \(\square\)