Problem 1  Give the definition of each part below.

(a) A proposition.
   Answer: A proposition is a sentence that is either true or false.

(b) The equivalence of two open sentences.
   Answer: In a given universe, two open sentences are equivalent iff they have the same truth set.

(c) The intersection of two sets $A$ and $B$.
   Answer: The intersection of $A$ and $B$ is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Problem 2  Show that the propositional forms $[P \Rightarrow (Q \lor R)]$ and $[(P \land \sim R) \Rightarrow Q]$ are equivalent.

Solution: From the table below we can see that $[P \Rightarrow (Q \lor R)]$ and $[(P \land \sim R) \Rightarrow Q]$ have the same truth tables, thus are equivalent.

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Problem 3  In each of the following cases, first translate each symbolic expression into ordinary English statement. Then negate each expression symbolically, pushing the negation symbol as far in as possible. Final, give a translation of each negated expression into ordinary English.

(a) \((\exists e)\{e > 0 \land (\forall d)[d > 0 \Rightarrow (\forall x)(0 < |x - a| < d \Rightarrow |f(x) - L| < e)]\}\), where \(d, e,\) and \(x\) are real numbers.

Solution: English: There exits some positive number \(e\) such that for all \(x \neq a\), the difference between \(f(x)\) and \(L\) is smaller than \(e\).

Negation: \((\forall e)\{e > 0 \Rightarrow (\exists d)[d > 0 \land (\exists x)(0 < |x - a| < d \land |f(x) - L| \geq e)]\}\). (The universe is \(\mathbb{R}\).)

English: For all positive number \(e\), there exists some positive \(d\) such that for some \(x\) in the interval \((a - d, a + d)\) the difference between \(f(x)\) and \(L\) is no less than \(e\).

(b) \((\forall a)(\forall b)(\forall c)(a \text{ divides } bc \Rightarrow (a \text{ divides } b \lor a \text{ divides } c))\), where \(a, b, c\) are integers.

Solution: English: For any integers \(a, b, c\), if \(a\) divides \(bc\), then \(a\) divides \(b\) or \(a\) divides \(c\).

Negation: \((\exists a)(\exists b)(\exists c)(a \text{ divides } bc \land a \text{ does not divide } b \land a \text{ does not divide } c)\). (The universe is \(\mathbb{Z}\).)

English: There exist integers \(a, b, c\), such that \(a\) divides \(bc\) but \(a\) divides neither \(b\) nor \(c\).
Problem 4  Determine whether each statement is true or false. If a statement is true, prove it. If it is false, give a counterexample.

(a) If \( x \) is an even integer and \( y \) is a multiple of \( x \) (by an integer), then \( y \) is even.

Answer: The statement is true.

Proof: Let \( x \) be an even integer and \( y \) is a multiple of \( x \).

By the definition of even, \( x = 2k \), for some integer \( k \). Since \( y \) is a multiple of \( x \), there exists an integer \( m \) such that \( y = mx \). Therefore,

\[
y = mx = m(2k) = 2(mk).
\]

Because \( mk \) is an integer, \( y = 2(mk) \) is even.

Thus, if \( x \) is an even integer and \( y \) is a multiple of \( x \) (by an integer), then \( y \) is even. \( \square \)

(b) If \( t^2 = 9 \), then \( t > 2 \).

Answer: The statement is false.

Counterexample: Let \( t = -3 \). we still have that \( t^2 = 9 \). However, \( -3 < 2 \).

(c) Let \( x, y \) and \( z \) be integers. If \( xy \) is odd, then both \( x \) and \( y \) are odd.

Answer: The statement is true.

Proof: Let \( x, y \) and \( z \) be integers. We will prove by contraposition.

Suppose \( x \) or \( y \) is not odd. Then (at least) one of them is even. Without loss of generality, we assume \( x \) is even. Then by the definition of even, \( x = 2k \) for some integer \( k \). Thus, \( xy = (2k)y = 2(ky) \). Since \( ky \) is an integer, we must have that \( xy \) is even, which is not odd.

Therefore, if \( x \) or \( y \) is not odd, then \( xy \) is not odd.

Thus, by contraposition, if \( xy \) is odd, then both \( x \) and \( y \) are odd. \( \square \)
Problem 5

(a) Write the tautology that justifies the proof of a conditional proposition by contradiction. (Hint: Instead of proving \( P \Rightarrow Q \), what is the equivalent propositional form we show?)

\[ (P \Rightarrow Q) \iff \sim (P \Rightarrow Q) \Rightarrow (R \land \sim R) \]

\[ \text{Or: } (P \Rightarrow Q) \iff [(P \land \sim Q) \Rightarrow (R \land \sim R)] \]

(b) Prove the following by contradiction: Let \( x \) be an integer. If \( x^2 \) is even, then \( x \) is even.

**Proof by contradiction:** Let \( x \) be an integer. Suppose \( x^2 \) is even and \( x \) is odd.

By the definition of odd, \( x = 2k + 1 \), for some integer \( k \). Then

\[ x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1. \]

\( k \in \mathbb{Z} \) implies that \( 2k^2 + 2k \in \mathbb{Z} \). Hence, \( x^2 \) is odd, which is a contradiction to \( x^2 \) is even.

Therefore, if \( x^2 \) is even, then \( x \) is even. \( \square \)

Problem 6

Let the universe be all real numbers \( \mathbb{R} \). For any \( B \subseteq \mathbb{R} \), define \( B^* = B \cup \{0\} \).

Prove that \( B = B^* \) iff \( 0 \in B \).

**Proof:** We will use two-part proof of a biconditional sentence. Let \( B \subseteq \mathbb{R} \) and \( B^* = B \cup \{0\} \).

(a) Suppose \( B = B^* \). Then \( 0 \in B \cup \{0\} = B^* = B \).

Therefore, if \( B = B^* \), then \( 0 \in B \).

(b) Suppose \( 0 \in B \). We will show \( B = B^* \) by proving \( B \subseteq B^* \) and \( B^* \subseteq B \).

First, \( B \subseteq B \cup \{0\} = B^* \), because \( B \subseteq B \cup A \), for any set \( A \).

Next, \( 0 \in B \) implies that \( \{0\} \subseteq B \). Unioning \( B \) on both sides of \( \{0\} \subseteq B \) gives \( B \cup \{0\} \subseteq B \cup B \). However, \( B \cup \{0\} = B^* \) and \( B \cup B = B \). Thus, we have \( B^* \subseteq B \).

We showed that \( B \subseteq B^* \) and \( B^* \subseteq B \). Thus, we conclude that \( B = B^* \).

Therefore, if \( 0 \in B \), then \( B = B^* \).

Therefore, \( B = B^* \) iff \( 0 \in B \). \( \square \)
Problem 7  Let $A$, $B$ and $C$ be sets. Indicate if this statement is true or false and prove your answer:

(a) $(A - B) \cup (A - C) = A - (B \cup C)$.

**Answer:** The statement is false. (You can use a Venn diagram to convince yourself. But a Venn diagram cannot serve as a proof. We give a counterexample.)

**Counterexample:** Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, and $C = \{3, 4, 5\}$.

Then $A - B = \{1\}$, $A - C = \{1, 2\}$, and so $(A - B) \cup (A - C) = \{1, 2\}$.

$B \cup C = \{2, 3, 4, 5\}$ and $A - (B \cup C) = \{1\}$.

Hence, in this example, $(A - B) \cup (A - C) \neq A - (B \cup C)$.

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

**Answer:** This statement is true.

**Proof:** Let $A, B$ and $C$ be sets. We show that for any $(x, y)$, $(x, y) \in A \times (B \cup C)$ iff $(x, y) \in (A \times B) \cup (A \times C)$. Let $(x, y)$ be an ordered pair of objects. Then

$(x, y) \in A \times (B \cup C) \iff x \in A$ and $y \in B \cup C$

iff $x \in A$ and $(y \in B$ or $y \in C)$

iff $(x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

iff $(x, y) \in A \times B$ or $(x, y) \in A \times C$

iff $(x, y) \in (A \times B) \cup (A \times C)$.

Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$. \qed

(c) If $A \cap B \cap C = \emptyset$, then $A \cap B = A \cap C = B \cap C = \emptyset$.

**Answer:** The statement is false.

**Counterexample:** Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{1, 3\}$. Then $A \cap B \cap C = \emptyset$.

But $A \cap B = \{1, 2\} \cap \{2, 3\} = \{2\} \neq \emptyset$. 

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