BASIC ASSUMPTIONS

1. LAWS OF ELEMENTARY ALGEBRA

1.1. Properties of operations. Suppose \( x, y, z \) are integer numbers, real numbers or complex numbers.

(1) The operation of addition +:
   (a) \((x + y) + z = x + (y + z)\).
   (b) \(x + 0 = 0 + x = x\).
   (c) for every \( x \), there is an additive inverse \(-x\) such that \( x + (-x) = (-x) + x = 0\).
   (d) \(x + y = y + x\).

(2) The operation of multiplication \(\cdot\):
   (a) \((x \cdot y) \cdot z = x \cdot (y \cdot z)\).
   (b) \(x \cdot 1 = 1 \cdot x = x\).
   (c) for every \( x \neq 0 \), there must be a multiplicative inverse \( x^{-1} \) such that \( x \cdot x^{-1} = x^{-1} \cdot x = 1\).
   (d) \(x \cdot y = y \cdot x\).

(3) \( x \cdot (y + z) = x \cdot y + x \cdot z\).
(4) Each of the following sets is closed under both addition + and multiplication \(\cdot\):
   - the integer set \(\mathbb{Z}\), the real number set \(\mathbb{R}\), and the complex number set \(\mathbb{C}\).

   This says that if \( x \) and \( y \) are in \(\mathbb{Z}\), then both \( x + y \) are \( x \cdot y \) in \(\mathbb{Z}\). Same statement holds for \(\mathbb{R}\) and \(\mathbb{C}\).

1.2. Laws of inequality. Suppose \( x, y, z \) are real numbers.

(1) \( x \neq x \) (irreflexivity).
(2) If \( x < y \) and \( y < z \), then \( x < z \) (transitivity).
(3) Exactly one of \( x < y \), \( x = y \), or \( y < x \) is true.
(4) If \( x < y \), then \( x + z < y + z \).
(5) If \( x < y \) and \( 0 < z \), then \( xz < yz \).
(6) If \( x < y \) and \( z < 0 \), then \( xz > yz \).

2. Basic definitions

(1) A real number \( x \) is \textbf{positive} if and only if \( x > 0 \). A positive integer is also called a \textbf{natural number}.
(2) A real number \( x \) is \textbf{negative} if and only if \( x < 0 \).
(3) An integer \( x \) is \textbf{even} if and only if there is an integer \( k \) such that \( x = 2k \).
(4) An integer \( x \) is \textbf{odd} if and only if there is an integer \( j \) such that \( x = 2j + 1 \).
(5) For integers \( a \) and \( b \) where \( a \neq 0 \), we say \( a \) \textbf{divides} \( b \) if and only if there is an integer \( k \) such that \( b = ak \).
(6) Two integers $x$ and $y$ have a **common factor** if and only if there exists a positive integer $k > 1$ such that $k$ divides both $x$ and $y$.

(7) A positive integer $p$ is **prime** if and only if $p$ is greater than 1 and the only positive integers that divide $p$ are 1 and $p$.

(8) The real number $x$ is **rational** if and only if there exist integers $p$ and $q$, where $q \neq 0$, such that $x = p/q$.

(9) The **absolute value** of a real number $x$, denoted by $|x|$, is defined by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

**Note.** After we covered Section 1.4, you may assume simple facts about absolute values such as $|x| \geq 0$ and the results stated in Exercises 1.4.6, unless a problem asks you to prove one of these results.

### 3. Basic results from number theory

**Theorem.** The positive integer 1 is the least positive integer.

**Theorem** (Fundamental Theorem of Arithmetic). Every positive integer larger than 1 can be expressed uniquely as a product of primes.

**Lemma 1.** Any integer is either an even number or an odd number, but cannot be both.

**Lemma 2.** Two integers $x$ and $y$ have a common factor if and only if they have a common **prime** factor.

**Lemma 3.** Any rational number can be written as $p/q$ where $p$ and $q \neq 0$ are integers satisfying that $p$ and $q$ do not have common factors.

### 4. Fundamental results from Calculus

**Theorem** (Intermediate Value Theorem). If $f(x)$ is a continuous function on the closed interval $[a, b]$, then for any number $z$ between $f(a)$ and $f(b)$, there exists a number $c \in [a, b]$ such that $f(c) = z$.

**Theorem** (Fundamental Theorem of Algebra). Suppose $f(x)$ is a polynomial of degree $d \geq 1$. If the coefficients of $f$ are all complex numbers, then $f$ has at least one complex root.

**Corollary.** Suppose $f(x)$ is a polynomial of degree $d \geq 1$. If the coefficients of $f$ are all complex numbers, then $f$ has exactly $d$ complex roots (counting multiplicities).

**Theorem** (Complex Conjugate Root Theorem). Suppose $f(x)$ is a polynomial with real coefficients. If $a + bi$ (for some real numbers $a, b$) is a root of $f$, then so is its complex conjugate $a - bi$.

Therefore, the number of non-real roots of $f$ is even.