BASIC ASSUMPTIONS

1. LAWS OF ELEMENTARY ALGEBRA

1.1. Properties of operations. Suppose \( x, y, z \) are integer numbers, real numbers or complex numbers.

(1) The operation of addition +:
   (a) \( (x + y) + z = x + (y + z) \).
   (b) \( x + 0 = 0 + x = x \).
   (c) for every \( x \), there is an additive inverse \( -x \) such that \( x + (-x) = (-x) + x = 0 \).
   (d) \( x + y = y + x \).

(2) The operation of multiplication \( \cdot \):
   (a) \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \).
   (b) \( x \cdot 1 = 1 \cdot x = x \).
   (c) for every \( x \neq 0 \), there must be a multiplicative inverse \( x^{-1} \neq 0 \) such that \( x \cdot x^{-1} = x^{-1} \cdot x = 1 \).
   (d) \( x \cdot y = y \cdot x \).

(3) \( x \cdot (y + z) = x \cdot y + x \cdot z \).

(4) Each of the following sets is closed under both addition + and multiplication \( \cdot \):
   - the integer set \( \mathbb{Z} \), the real number set \( \mathbb{R} \), and the complex number set \( \mathbb{C} \).

   This says that if \( x \) and \( y \) are in \( \mathbb{Z} \), then both \( x + y \) are \( x \cdot y \) in \( \mathbb{Z} \). Same statement holds for \( \mathbb{R} \) and \( \mathbb{C} \).

1.2. Laws of inequality. Suppose \( x, y, z \) are real numbers.

(1) \( x \not< x \) (irreflexivity).

(2) If \( x < y \) and \( y < z \), then \( x < z \) (transitivity).

(3) Exactly one of \( x < y \), \( x = y \), or \( y < x \) is true.

(4) If \( x < y \), then \( x + z < y + z \).

(5) If \( x < y \) and \( 0 < z \), then \( xz < yz \).

(6) If \( x < y \) and \( z < 0 \), then \( xz > yz \).

2. BASIC DEFINITIONS

(1) A real number \( x \) is positive if and only if \( x > 0 \). A positive integer is also called a natural number.

(2) A real number \( x \) is negative if and only if \( x < 0 \).

(3) An integer \( x \) is even if and only if there is an integer \( k \) such that \( x = 2k \).

(4) An integer \( x \) is odd if and only if there is an integer \( j \) such that \( x = 2j + 1 \).

(5) For positive integers \( a \) and \( b \), we say \( a \) divides \( b \) if and only if there is a positive integer \( k \) such that \( b = ak \).
(6) Two integers $x$ and $y$ have a **common factor** if and only if there exists a positive integer $k > 1$ such that $k$ divides both $x$ and $y$.
(7) A positive integer $p$ is **prime** if and only if $p$ is greater than 1 and the only positive integers that divide $p$ are 1 and $p$.
(8) The real number $x$ is **rational** if and only if there exist integers $p$ and $q$, where $q \neq 0$, such that $x = p/q$.

3. Basic results from number theory

**Theorem** (Fundamental Theorem of Arithmetic). Every positive integer larger than 1 can be expressed uniquely as a product of primes.

**Lemma 1.** Two integers $x$ and $y$ have a common factor if and only if they have a common prime factor.

**Lemma 2.** Any rational number can be written as $p/q$ where $p$ and $q \neq 0$ are integers satisfying that $p$ and $q$ do not have common factors.