SUPPLEMENTAL NOTES ON CHAPTER 5

Unlike the other chapters, most of the proofs you write for this chapter are not just based on definitions; you often need to apply theorems. Here is a summary of the proof methods related to the concepts we learn in Chapter 5.

1. Basic methods of proving $A \approx B$:
   - The most common way is directly showing there exists a bijection $f : A \to B$. Usually first construct a function $f : A \to B$ then show $f$ is a bijection by using the methods we learned in chapter 4.
   - Sometimes use Lemma 5.1.2.

2. Basic methods of proving a set is finite:
   - Directly verify the definition.
   - Use Theorem 5.1.3: If $B \approx A$ and $A$ is finite, then $B$ is finite.
   - Use Theorem 5.1.6: If $B \subseteq A$ and $A$ is finite, then $B$ is finite.
   - Use Theorem 5.1.7: Union of finitely many finite sets is finite.

3. Basic methods of proving a set is infinite:
   - Suppose $S$ is finite. Then try to get a contradiction. Two common way of finding a contradiction:
     - Use theorems on finite sets;
     - Show the bijection between $S$ and $\mathbb{N}_k$ is either not injective or not surjective.
   - Prove $S$ is equivalent to some known infinite set.
   - Use an equivalent form of Theorem 5.1.6: If $B \subseteq A$ and $B$ is infinite, then $A$ is infinite.
   - Use the contrapositive of Corollary 5.1.11: If $S$ is equivalent to one of its proper subset, then $S$ is infinite.

4. Basic methods of proving a set is denumerable:
   - Directly prove $S \approx \mathbb{N}$ by constructing a one-to-one correspondence between $S$ and $\mathbb{N}$.
   - Prove $S$ is equivalent to some known denumerable set: $\mathbb{N}, \mathbb{E}^+, \mathbb{Z}, \mathbb{N} \times \mathbb{N}, \mathbb{Q}^+, \mathbb{Q}$.
   - Use Theorem 5.2.3/(b): If $A$ and $B$ are denumerable, then $A \times B$ is denumerable.
   - Use Theorem 5.3.5: If $A$ is denumerable and $B$ is finite, then $A \cup B$ is denumerable.
   - Use a consequence of Theorem 5.3.6: Union of finitely many pairwise disjoint denumerable sets is denumerable.
   - Prove $S$ is countable and infinite using methods listed in (3) and (5).
(5) Basic methods of proving a set is countable:
   (a) Prove $S$ is either finite or denumerable by using the methods listed in (2) or (4).
   (b) Use Theorem 5.3.2: If $B \subseteq A$ and $A$ is countable, then $B$ is countable.
   (c) Use Theorem 5.3.8 (and its corollary): Union of a countable collection of countable sets is countable.

(6) Basic methods of proving a set is uncountable:
   (a) Prove by contradiction. Assume the set is countable. Then apply theorems on countable sets to find a contradiction.
   (b) Prove $S$ is infinite and not denumerable: See (3) for methods of proving infinite sets. Prove not denumerable by contradiction.
   (c) Prove $S$ has cardinality $\mathfrak{c}$: See (7).

(7) Basic methods of proving a set has cardinality $\mathfrak{c}$:
   (a) Prove $S$ is equivalent to $(0, 1)$.
   (b) Prove $S$ is equivalent to some set known to have cardinality $\mathfrak{c}$: $(a, b)$, $(a, +\infty)$, $(-\infty, b)$, $\mathbb{R}$.

Finally, the Cantor-Schroder-Berstein Theorem (Theorem 5.4.4) provides us another way to prove a set has a certain cardinal number especially the infinite ones (e.g. $\mathbb{N}_0, \mathfrak{c}$). For example, to prove a set $S$ has cardinal number $\mathfrak{c}$, instead of constructing a bijection from $S$ to a known set with cardinality $\mathfrak{c}$, you can find two sets $A$ and $B$ of cardinality $\mathfrak{c}$, and then show that there are injections from $A$ to $S$ and $S$ to $B$. 