Appendix A

PROOF PRACTICE

Throughout this appendix, you are allowed to use basic high school algebra, plane geometry, and properties of numbers that you have come to take for granted. The goal is to give you some focused practice

- on the process to proof and
- on completing direct, contrapositive, and by contradiction proofs.

A brief review of the process to proof is followed by some examples.

Keeping the following process in mind can help you towards being able to produce proofs. The descriptions of the three phases are offered to suggest the types of experiences that are entailed; the more work you do towards proving the more you will discover what works well for you.

**Phase I: Introduction to the Problem**

Before any attempt can be made at proving a statement, it is necessary to understand exactly what it is that is to be proved. After reading the problem carefully, think about it. In this phase, we make sure that we understand the terminology as well as what we are allowed to use in order to argue to a desired conclusion. It is also a good idea to try to “check out” the result with some particular examples. In addition, talking about the problem can help to clarify it and can lead to some valuable, shared, insight.

**Phase II: Scratch Work Towards a Proof**
APPENDIX A. PROOF PRACTICE

The few examples that might have been done as part of Phase I are not sufficient to claim a statement as proved. With this phase you actually manipulate abstract objects in an effort to find something that justifies what is claimed. You may need to introduce other concepts or draw upon other theorems or postulates. It is even acceptable—at this point—to try to work backwards from the conclusion. With this last possibility, if the steps are reversible, your proof can be a well written presentation of the steps—starting with the hypothesis (of course) and leading to the conclusion. Your scratch work is for your eyes and your benefit only.

Do your scratch work on a separate sheet with the expectation that it is for your future reference only. Use your scratch work sheet(s) to work out all the details that are needed in order to justify a given claim and to determine the order in which you want to present the steps of a proof. During this part, you will decide on which method of proof (direct, indirect, by contrapositive, by induction, etc.) will lead to the neatest presentation. Even deciding on the method of proof can take a bit of trial and error.

Once you know the basics of the solution, you are ready to translate your preparatory work into a solution that meets criteria for good exposition. This leads to the last phase.

Phase III: Communicating a Proof

Once you believe that you have a convincing argument, the process is concluded with a careful, well written presentation of that argument. The usual rules for writing good English are in effect for this phase. In addition, every attempt should be made to maximize neatness. Remember that sloppy reporting is evidence of sloppy thinking.

A.1 Some Examples

1. Prove that \( 1 - (\sec \alpha)(\cos \alpha) = (\tan \alpha)(\cot \alpha) - 1 \) where \( \alpha \) is the measure of an angle given in radians and \( \alpha \) differs from \( k\pi \) and \( (\pi/2) + k\pi \) for \( k \in \mathbb{Z} \), \( \mathbb{Z} \) denotes the set of integers.

   In Phase I, we make sure that we know the definitions of secant, cosine, tangent and cotangent. We might also try the value \( \pi/4 \) for \( \alpha \).
In Phase II, we would most likely go back to the method that we used in high school to verify trig identities. That is, the scratch work would use substitution and simplification on both sides of the given identity until we arrive at the same thing on both sides of the equals sign. For example:

\[ 1 - (\sec \alpha) \cos \alpha = (\tan \alpha) (\cot \alpha) - 1 \]

\[ 1 - \left( \frac{1}{\cos \alpha} \right) \cos \alpha = \left( \frac{\sin \alpha}{\cos \alpha} \right) \left( \frac{\cos \alpha}{\sin \alpha} \right) - 1 \]

\[ 1 - 1 = 1 - 1 \]

\[ 0 = 0. \]

Now we know that the statement is true and we are ready to present a proof. Note that this scratch work is definitely not a proof for two reasons. (a) The sequence of steps starts with the conclusion and A PROOF NEVER STARTS WITH THE CONCLUSION. (b) There are problems with the exposition. For one thing, the sentences started with numbers. In addition, the identity of \( \alpha \) needs to be specified.

Phase III consists of writing up the proof that is to be offered.

Proof. Let \( \alpha \) be the measure of an angle given in radians such that \( \alpha \neq k\pi \) and \( \alpha \neq (\pi/2) + k\pi \) for \( k \in \mathbb{Z} \). Since \( \alpha \neq (\pi/2) + k\pi \) for \( k \in \mathbb{Z} \), we know that the \( \sec \alpha \) is defined. From the restriction that \( \alpha \neq k\pi \) for \( k \in \mathbb{Z} \), we also have that \( \sin \alpha \neq 0 \). Thus, we deduce that

\[ 1 - (\sec \alpha) (\cos \alpha) = 1 - \left( \frac{1}{\cos \alpha} \right) (\cos \alpha) \]

\[ = 1 - 1 \]

\[ = \left( \frac{1}{\cos \alpha} \right) (\cos \alpha) - 1 \]

\[ = \left( \frac{\sin \alpha}{\cos \alpha} \right) \left( \frac{\cos \alpha}{\sin \alpha} \right) - 1 \]

\[ = (\tan \alpha) (\cot \alpha) - 1. \]
Therefore, for $\alpha$ the measure of an angle given in radians with $\alpha \neq k\pi$ and $\alpha \neq \left(\pi/2\right) + k\pi$ for $k \in \mathbb{Z}$, we have that $1 - \left(\sec \alpha\right)\left(\cos \alpha\right) = 1 - \left(\tan \alpha\right)\left(\cot \alpha\right)$. 

2. Prove that there is exactly one pair of primes $p$ and $q$ such that $q - p = 3$.

In Phase I, we make sure that we know what a prime is and try to look at some examples. In particular, we should recall that 2 is the only even prime.

For Phase II, we only need to rough out the details that just dropped out of Phase I. Since all but one of the primes is odd and the difference between two odd numbers is even, we know that one of the primes in the pair we are looking for must be the number 2. That gives us only one possible pair.

Phase III. Proof. Let $p$ and $q$ be primes. If $q - p = 3$, then one of the primes must be even and $p < q$. Hence, $p = 2$ is the only possibility. Since $5 - 2 = 3$ and 5 is a prime, we conclude that $(2, 5)$ is the only pair of primes for which the difference between them is 3. Therefore, there is exactly one pair of primes, $p$ and $q$, such that $q - p = 3$. 

Excursion A.1.1 After filling in your own ideas for the three phases, compare what you did with the discussion that follows. Prove that the line $\ell_1$ given by $2x + 3y = 2$ is parallel to the line $\ell_2$ given by $y = -\frac{2}{3}x + 5$.

PHASE I (Introduction to the problem.)

PHASE II (Scratch work towards a proof.)

PHASE III (Communicating a proof.)
A.2. SOME DIRECT PROOFS

Discussion of Excursion A.1.1

For Phase I, you just needed to note how one can tell that lines are parallel from equations that are given for them. You should have annotated that the slopes need to be the same and the slope is \( m \) when the equation is in the form \( y = mx + b \).

The scratch work (Phase II) should have put both of the given equations in the correct form so that the slopes could be compared. Voilà! Finding what we want leads us immediately to Phase III.

**Proof.** Recall that lines are parallel if and only if they have the same slopes. Also, for a line given in the form \( y = mx + b \), the \( m \) is the slope.

The line \( \ell_1 \) is given by \( 2x + 3y = 2 \). But

\[
2x + 3y = 2 \iff 3y = -2x + 2 \iff y = \frac{-2}{3}x + \frac{2}{3}.
\]

Thus, the slope of \( \ell_1 \) is \( \frac{-2}{3} \). The line \( \ell_2 \), which is given in slope–intercept form \( y = -\frac{2}{3}x + 5 \), has slope \( \frac{-2}{3} \). Since the lines \( \ell_1 \) and \( \ell_2 \) have the same slopes, we conclude that they are parallel. \( \blacksquare \)

**Remark A.1.1** Note that, in our version, the reader is told what the writer is using—without unnecessary observations—and the argument is presented in complete sentences. Also the symbol \( \iff \) has only been used between complete sentences.

**Remark A.1.2** When you read the proof, you should have read everything that was written. It is common for readers not to read the symbols or the equations: Such omissions can lead to errors in exposition because things that are written twice go unnoticed.

Before going on, compare your proof to the one that was given. Grade your proof for exposition as well as mathematical correctness. Check for sentence length and punctuation. Use the annotations that are given in Section B.2.4: Guidelines for Self Analysis.

A.2 SOME DIRECT PROOFS

For a direct proof of \( P \Rightarrow Q \), we start with “Suppose \( P \)” and argue to \( Q \). In the first excursion, the table provides a schema with which to work. It essentially outlines the form of a scratch work phase, when you know that you are in pursuit of a direct proof.
Excursion A.2.1 Fill in the table to complete the suggested scratch work for a solution to the following problem. Then offer a well presented proof. Comments and an acceptable proof follow the remark given below.

<table>
<thead>
<tr>
<th>* The set-up</th>
<th>Assume that $a$ and $b$ are odd integers.</th>
</tr>
</thead>
<tbody>
<tr>
<td>* Pull in Definitions or translations</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>* Deductive Reasoning</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>* Closure or statement of what was proved</td>
<td></td>
</tr>
</tbody>
</table>

Proof.

Remark A.2.1 Excursion A.2.1 is as straightforward as a proof can get. In practice, you might have several times that you work through such a basic structure, having to introduce new objects at various stages and calling upon earlier results.
A.2. SOME DIRECT PROOFS

Discussion of Excursion A.2.1

For the scratch work, use of the definitions gave us the chance to claim the existence of two integers, say \( j \) and \( k \), such that \( a = 2j + 1 \) and \( b = 2k + 1 \). Notice that you had to use two different letters for your claimed integers; otherwise, you would have been adding the condition that \( a = b \) which is not given.

For the deductive reasoning, the problem tells us to consider the product. We use the new form for \( a \) and \( b \) to get a form for the product. This leads us to \( ab = 4jk + 2j + 2k + 1 \). We want to be able to declare that \( ab \) is odd, we will be able to do this if we can write it in the form of \( 2m + 1 \) for some integer \( m \). Seeing this enables us to bring the deductive reasoning home: We can factor a 2 out of the first three terms in the expression that we obtained for \( ab \) in terms of \( j \) and \( k \). Since \( 2jk + j + k \) is an integer, we are done with the scratch work.

**Proof.** Suppose that \( a \) and \( b \) are odd integers. Then there exists integers \( j \) and \( k \) such that \( a = 2j + 1 \) and \( b = 2k + 1 \). It follows that

\[
ab = 4jk + 2j + 2k + 1 = 2(2jk + j + k) + 1.
\]

Now \( j, k \in \mathbb{Z} \) implies that \( 2jk + j + k \in \mathbb{Z} \), and we conclude that \( ab \) is an odd integer. Since \( a \) and \( b \) were arbitrary, we have that the product of any two odd integers is an odd integer. \( \blacksquare \)

Take the time to compare what you did towards completing the excursion with the discussion that was offered. Look, with a very critical eye, to see where you did things differently and where some improvements could have been made. After you are satisfied with your self assessment and analysis, work on the following.

**Excursion A.2.2** Give a direct proof for the following claim.

*If \( x \) is an odd integer, then \( 4x^2 + 3x + 6 \) is an odd integer.*
Space for Phases I and II:

**Proof.** (Phase III)

**Discussion of Excursion A.2.2**

This one should have been a confidence builder. Because you are told to pursue a direct proof, you are spared any need to ruminate over how to proceed. For Phase I, you should have claimed that since \( x \) an odd integer, there exists an integer \( j \) such that \( x = (2j + 1) \). To find out if \( 4x^2 + 3x + 6 \) is even or odd, we substitute and simplify. Upon seeing that everything works, we are ready to present a well written, concise proof of the claim.

**Proof.** Let \( x \) be an odd integer. Then there exists an integer \( j \) such that \( x = 2j + 1 \). It follows that

\[
4x^2 + 3x + 6 = 4(2j + 1)^2 + 3(2j + 1) + 6 = 2 \left( 8j^2 + 11j + 6 \right) + 1.
\]

Since \( (8j^2 + 11j + 6) \) is an integer, whenever \( j \) is an integer, we conclude that \( 4x^2 + 3x + 6 \) is an odd integer. Therefore, if \( x \) is an odd integer, then \( 4x^2 + 3x + 6 \) is an odd integer. \( \square \)

**Remark A.2.2** Notice that, in our proof, the reader was spared having to sift through all the algebra that got to the final form for \( 4x^2 + 3x + 6 \) when \( x = 2j + 1 \). The essential information is what was shared.

**A.3 Some Proofs by Contrapositive**

For a proof by contrapositive of a conditional statement we offer a direct proof of the contrapositive of the statement. Consequently, to prove \( P \Rightarrow Q \) by contrapositive,
we start by assuming that \( \neg Q \) is true and form an argument to show that \( \neg P \) is true. Then we observe that we have shown \( \neg Q \Rightarrow \neg P \) is true which is equivalent to \( P \Rightarrow Q \) being true.

**Excursion A.3.1** For the following result, a sequence of conditional statements that fit together for a proof by contrapositive is given on the left. Use the space on the right, to offer a well-written, lean proof (Phase III).

**Over the natural numbers, \( n \) is even whenever \( n^2 \) is even.**

**Proof**

**Final Part of Scratch Work.** Suppose that \( n \) is a natural number. If \( n \) is not even, then \( n \) is odd. If \( n \) is odd, then we can write \( n = 2k + 1 \) for some natural number \( k \). If \( n = 2k + 1 \) for some natural number \( k \), then \( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 \), for some natural number \( k \). If \( n^2 = 4k^2 + 4k + 1 \) for some natural number \( k \), then \( n^2 = 2(2k^2 + 2k) + 1 \) for some natural number \( k \). If \( n^2 = 2(2k^2 - 2k) + 1 \) for some natural number \( k \), then \( n^2 \) is odd because \( 2k^2 - 2k \) is a natural number. If \( n^2 \) is odd, then \( n^2 \) is not even. Therefore, we have shown that if \( n \) is not even, then \( n^2 \) is not even. By contraposition, we conclude that if \( n^2 \) is even, then \( n \) is even.

**Discussion of Excursion A.3.1**

What was offered for the final part of scratch work was more elaborate and detailed than scratch work you might have produced. It is offered here to remind you of the role that the transitivity tautology plays in the deductive process. Your proof just needed to clean up repeated phrases and modify to suit personal writing style—within our specified guidelines. Also, you needed to find an omitted observation that had to be inserted in order to get exactly what was claimed.

Compare your proof with the following offering. Did you find the observation that needed to be made though it was omitted in the scratch work? If not, can you find it in what is offered below? Try grading your version (or ours) using the annotations that were given in Section B.2.4: Guidelines for Self Analysis.
Proof. Suppose that \( n \) is a natural number that is not even. Then \( n \) is odd and there exists a natural number \( k \) such that \( n = 2k - 1 \). It follows that

\[
\begin{align*}
n^2 = (2k - 1)^2 &= 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1. 
\end{align*}
\]

Since \( (2k^2 - 2k) = 2k(k - 1) \geq 0 \) for \( k \in \mathbb{N} \), we have that \( 2k^2 - 2k \) is a nonnegative integer. Thus, \( n^2 \) is an odd natural number. Hence, \( n^2 \) is not even. Therefore, we have shown that if \( n \) is not even, then \( n^2 \) is not even. By contraposition, we conclude that if \( n^2 \) is even, then \( n \) is even.

Excursion A.3.2 Give a proof by contrapositive of the following claim.

Suppose that \( x \) and \( y \) are integers. If \( x \cdot y \) is an even integer,

then \( x \) is an even integer or \( y \) is an even integer.

Space for Phases I and II:

Phase III:

Discussion of Excursion A.3.2

For Phase I, you should have written down the contrapositive of what you want to show, since it gives us the set-up for the approach to proof that we are told to pursue. From DeMorgan’s tautology, we find that we need to start with two integers, both of which are not even. Since being not even is equivalent to being odd, we go directly to the definition to see whether the product of the integers is even or odd. Your scratch work would have two integers, say \( j \) and \( k \) such that \( x = 2j + 1 \) and \( y = 2k + 1 \). Then \( x \cdot y \) becomes \( (2j + 1)(2k + 1) = 4jk + 2j + 2k + 1 = 2(2jk + j + k) + 1 \). But hold on a minute, this looks awfully familiar, Ah! Ha!, we already showed that the product of two odd integers is odd. Because the result is in this section, we can cite it. Otherwise, you would have to finish the argument.
We will offer both proofs. Remember that we can use the first one, only because we have proved the needed result in this appendix.

**Proof.** (citing our result from another section) Suppose that \( x \) and \( y \) are integers such that \( x \) is not even and \( y \) is not even. Then \( x \) and \( y \) are odd integers. By Excursion A.2.1, we know that the product \( x \cdot y \) is an odd integer. Therefore, \( x \cdot y \) is not even. Hence, we have shown that if \( x \) and \( y \) are integers that are not even, then their product is not even. By contraposition, this is equivalent to the claim that if \( x \) and \( y \) are integers such that \( x \cdot y \) is even, then \( x \) is even or \( y \) is even. □

**Proof.** (if having to do it from scratch) Suppose that \( x \) and \( y \) are integers such that \( x \) is not even and \( y \) is not even. Then \( x \) and \( y \) are odd integers and there exist integers \( j \) and \( k \) such that \( x = 2j + 1 \) and \( y = 2k + 1 \). It follows that

\[
x \cdot y = (2j + 1)(2k + 1) = 2(2jk + j + k) + 1.
\]

Since \( j, k \in \mathbb{Z} \) implies that \( (2jk + j + k) \in \mathbb{Z} \), we conclude that \( x \cdot y \) is odd; i.e., \( x \cdot y \) is not even. Hence, we have shown that if \( x \) and \( y \) are integers that are not even, then their product is not even. By contraposition, this is equivalent to the claim that if \( x \) and \( y \) are integers such that \( x \cdot y \) is even, then \( x \) is even or \( y \) is even. □

### A.4 Some Proofs by Contradiction

For a proof by contradiction of a proposition \( M \), we start with \( \neg M \), but we do not know the specific conclusion to seek: We know only that we are after a mathematical absurdity—a contradiction \( L \land \neg L \) for some proposition \( L \).

**Excursion A.4.1** Prove that, over the reals, \( \frac{x^2}{x^2 + 4} < 1 \).

NOTE: Once the method of proof has been indicated, the first sentence (or so) of the proof is essentially specified.
Discussion of Excursion A.4.1

Once you start with the correct “suppose”, it doesn’t take too long to run into a problem. Since we are assuming basic properties of natural numbers, we have that $4 > 0$. This is what our scratch work contradicted.

Compare your proof with the following offering.

**Proof.** Over the set of reals, suppose that $\frac{x^2}{x^2 + 4} \not\leq 1$. Hence, $\frac{x^2}{x^2 + 4} \geq 1$. Since the expression on the left is rational, we know that it is greater than or equal to one if and only if the numerator is greater than or equal to the denominator. Thus, we must have that $x^2 \geq x^2 + 4$.

From the additive inverse property, this implies that $0 \geq 4$, which is a mathematical absurdity. We conclude that the negation of our assumption must hold; i.e., $\frac{x^2}{x^2 + 4} < 1$, as claimed. ■

Our next example is one of the most celebrated uses of a proof by contradiction. Though you have probably seen it before we include it here for completeness and in the hope that your new found understanding of this approach to proof will give you an increased appreciation for this beautiful piece of mathematics which is thousands of years old.

**Excursion A.4.2** Fill in what is missing in order to complete the following proof that the $\sqrt{2}$ is irrational.
A.5. SOME MORE EXAMPLES

Proof. Suppose that \( \sqrt{2} = \frac{p}{q} \). Then there exist integers \( p \) and \( q \), having no factors in common, such that
\[
\sqrt{2} = \frac{p}{q} \quad \text{or} \quad q \cdot \sqrt{2} = p.
\]
Hence, \( 2 \cdot q^2 = p^2 \) and we conclude that \( p^2 \) is \( \) \( \). From Excursion \( \) \( \), it follows that \( p \) is even. Thus, there exists an integer \( \ell \) such that \( \) \( \).

(5) Complete the proof.

Discussion of Excursion A.4.2

For the short fill-ins, we have (1) \( \sqrt{2} \) is rational, (2) even, (3) A.3.1, and (4) \( p = 2\ell \). For (5), your argument should have led you to
\[
\left( p^2 = 4\ell^2 \land 2q^2 = p^2 \right) \Rightarrow q^2 = 2\ell^2.
\]
Then the conclusion that \( q \) is even allows us to claim a contradiction to the fraction \( \frac{p}{q} \) being “in lowest terms.”

A.5 Some More Examples

Excursion A.5.1 Fill in what is missing in the following proof by contradiction of the claim:
\[
(\forall x)(\forall y)(x, y \in \mathbb{R} \Rightarrow (|x + y| \leq |x| + |y|)).
\]
Proof. Suppose that there exists two real numbers \(a\) and \(b\) such that

\[a + b\]

Since \((\forall x)(x \in \mathbb{R} \Rightarrow |x|^2 = x^2)\),

\[|a + b|^2 = \]

Now \(|a + b| > |a| + |b|\) and \(|a| + |b| \geq 0\) implies that

\[|a + b|^2 > (|a| + |b|)^2 = \]

Consequently,

\[a^2 + 2ab + b^2 > \]

which simplifies to

\[ab > |a| \cdot |b|\]

Squaring both sides and simplifying, we obtain that

\[a^2b^2 > \]

Thus \(> 0\). The last inequality is a contradiction. Therefore,

\[\neg (\exists a)(\exists b)(a, b \in \mathbb{R} \land (|a + b| > |a| + |b|))\]. From Theorem 2.3.10 and the Takes convention, we conclude that

\[\]

*** The fill-ins are (1) \(|a + b| > |a| + |b|\), (2) \(a^2 + 2ab + b^2\), (3) \(a^2 + 2|a| \cdot |b| + b^2\), (4) \(|a|^2 \cdot |b|^2\), (5) \(a^2b^2\), (6) \(0\), and (7) \((\forall x)(\forall y)(x, y \in \mathbb{R} \Rightarrow |x + y| \leq |x| + |y|)\).

*** The next example uses the following terminology and notation.
A.5. SOME MORE EXAMPLES

Notation A.5.1 Recall that, given two integers $a$ and $b$ we say that “$a$ divides $b$,” written $a | b$, if and only if

$$(\exists k) (k \in \mathbb{Z} \land b = ak).$$

Excursion A.5.2 For the following, discuss Phase I and Phase II directing your discussion towards a proof by contrapositive of the following claim. Then analyze the offered Phase III and fill in what is missing.

Suppose $x$ is a positive integer. If $x$ divides 5 and $x \neq 1$, then $x$ does not divide 3.

<table>
<thead>
<tr>
<th>Phase I.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Phase II.</th>
</tr>
</thead>
</table>

| Phase III. Suppose that the positive integer $x$ divides 3. Then $\frac{1}{2}x$ or $x = 3$. That is, $x = 1$ or $x \nmid 5$. Therefore, if $x$ divides 3, $x = 1$ or $x$ does not divide 5. Thus, by $\frac{1}{4}c^2$, if $x \neq 1$ and $x \mid 5$, then $x \nmid 3$. |

*** The fill–ins are (1) $x = 1$ and (2) contraposition. ***

Remark A.5.2 The main point being made with Excursion A.5.2 was the need to be precise about what it assumed and where you need to go with the proof.

Excursion A.5.3 For the following, fill in what is missing in the proof (Phase III) offered for the Claim: Given a right triangle whose hypotenuse has length $c$ and whose legs have lengths $a$ and $b$, the triangle is isosceles if and only if its area is $\frac{1}{4}c^2$. 
**Proof.** Suppose that we have a right triangle with hypotenuse having length $c$ and sides with lengths $a$ and $b$.

If the triangle is isosceles, then $\text{____________}$. Hence, the area of the triangle is given by $\text{____________}$.

From the Pythagorean Theorem we also have that $a^2 + b^2 = \text{__________}$.

It follows from the transitivity of equals that $2a^2 = \text{__________}$ or $a^2 = \text{__________}$. Now substitution yields that and the area of the triangle is $\frac{1}{2}a^2 = \text{__________} = \text{__________}$. Therefore, if the triangle is isosceles, then its area is $\frac{1}{4}c^2$.

Conversely, suppose that the area of the triangle is $\frac{1}{4}c^2$.

From the formula for the area of a triangle, it follows that $\frac{1}{4}c^2 = \text{__________}$. However, by the Pythagorean Theorem, we also have that $\frac{1}{4}c^2 = \text{__________}$. It follows that $\text{__________} = \frac{a^2 + b^2}{4} \iff 2ab = a^2 + b^2 \iff 0 = \text{__________} \iff 0 = \text{__________} \iff a = b$. Thus, if the area of the triangle is $\frac{1}{4}c^2$, then the triangle is isosceles.

Combining the two implications yields the claim. 

*** Check your fill-ins with the following: (1) $a = b$, (2) $a^2/2$, (3) $c^2$, (4) $c^2/2$, (5) $\frac{1}{2}(\frac{1}{2}c^2)$, (6) $c^2/4$, (7) $ab/2$, (8) $\frac{1}{4}(a^2 + b^2)$, (9) $\frac{1}{2}ab$, (10) $a^2 - 2ab + b^2$, (11) $a - b$ ***

**Remark A.5.3** In the previous proof of a biconditional statement, we needed to prove the two separate conditional sentences independently. This was because of the nature of the set-up and the timing for use of the Pythagorean Theorem. Sometimes it is possible to prove an if and only if statement by a sequence of steps that
connect ⇔’s. This usually involves combinations of definitions and equation manipulations. When this occurs try to avoid more that six ⇔’s without a break. More than that is tantamount to offering a sentence that is too long.

**Excursion A.5.4** (A Proof by Exhaustion) Sometimes your Phase II work will lead you to the discovery that you can reduce what needs to be shown to the consideration of a “practical” number of cases. (What is practical depends on whether you are doing them by hand or with the aid of a computer.) The approach that consists of examining every possible case out of a reasonable number of cases is called a **process of exhaustion**. Complete the following proof by a process of exhaustion for the claim:

If \( p \) is a prime different from 2,
then it can be written in the form \( 4k + 1 \) or \( 4k + 3 \) for some integer \( k \).

**Proof.** Suppose that \( p \) is a prime that is different from 2. If \( p \) is divided by 4, then we obtain a remainder of \( \)\hspace{1cm} \( (1) \)\hspace{1cm} \( \).

That is, we have one and only one of the following for fixed \( p \) and for \( k \) a nonnegative integer:

\[
p = 4k, \quad p = 4k + 1, \quad \text{or} \quad 4k + 3. \hspace{1cm} (2) \hspace{1cm} (3)
\]

If \( p = 4k \), then \( \)\hspace{1cm} \( (4) \)\hspace{1cm} \( (5) \) which implies that \( p \) is not a prime. Thus, if \( p \) is a prime, then \( p \neq 4k \). If \( p = 4k + 2 \), then \( \)\hspace{1cm} \( (6) \) and \( p \) is not a prime. Thus, if \( p \) is a prime, then \( p \neq 4k + 2 \).

Therefore, by the process of elimination, we conclude that, if \( p \) is a prime other than 2,

\[
\text{Space for scratch work.}\]

\[
(7) \hspace{1cm} \]

*** Compare your responses with the following: (1) 0, 1, 2, or 3, (2) \( 4k + 2 \), (3) \( 4k + 3 \), (4) \( 4 \), (5) \( p \), (6) \( 2 | p \) (or \( p \) is even), (7) \( p = 4k + 1 \) or \( p = 4k + 3 \) ***

**Excursion A.5.5** Prove that \( (\exists! x) \ (x \in \mathbb{R} \land x - 5 = \sqrt{x + 7}) \).
Phases I and II: Here, you are allowed to “work backwards” from the equation.

Phase III. Here we need to show two parts: (a) there is an $x$ that satisfies the given equation, and (b) no real numbers, other than the one exhibited, satisfy the given equation.

Proof.

Discussion of Excursion A.5.5

Working backwards lead you to a quadratic formula with roots $x = 2$ and $x = 9$. The first one was not valid; it was a root ‘created’ by having squared both sides of an equation which ignored sign restrictions. Armed with $x = 9$ as the desired value, we are ready to offer the proof. Compare your argument with the following argument.

Proof. Note that $\sqrt{9} + 7 = \sqrt{16} = 4 = 9 - 5$. Thus there exists an $x$, namely $x = 9$, such that $x - 5 = \sqrt{x + 7}$. Now suppose that $u$ is a real number such that $u - 5 = \sqrt{u + 7}$. Then

$$u^2 - 10u + 25 = u + 7 \iff u^2 - 11u + 18 = 0$$
$$\iff (u - 2)(u - 9) = 0$$
$$\iff u = 2 \lor u = 9.$$  

We already know that $u = 9$ is a solution. On the other hand, if $u = 2$, then $u - 5 = -3$ while $\sqrt{2} + 7 = \sqrt{9} = +3$. This shows that $u = 2$ is not a solution. Therefore the only solution to $x - 5 = \sqrt{x + 7}$ is $x = 9$. Hence

$$(\exists! x)(x \in \mathbb{R} \land x - 5 = \sqrt{x + 7}).$$
A.5. SOME MORE EXAMPLES

A.5.1 Exercises

1. For each of the following, offer well written proofs that make use of appropriate definitions, simple algebra, basic logic, and the method of proof that is specified.

(a) Give a Proof by Contradiction of the following statement.

For \( x \in \mathbb{R} \), \( \frac{x^2}{x^2 + 3} < 1 \).

(b) Give a Direct Proof of the following statement.

\((\forall m)(\forall n)(\forall p)(m, n, p \in \mathbb{N} \land p \mid n \land p \mid m \Rightarrow p \mid (m - n))\).

(c) Give a Proof by Contraposition of the following statement.

For \( x \in \mathbb{N} \), if \( 2 \mid (x^2 + 3x + 3) \), then \( 2 \mid x \).

2. For each of the following, offer well written proofs that make use of appropriate definitions, simple algebra, basic logic and the specified method of proof.

(a) Give a \textit{direct proof} of the following statement.

If \( x \equiv 3 \pmod{5} \), then \( 5 \mid (3x + 1) \).

(b) Give a \textit{proof by contrapositive} of the following statement.

For \( A \) and \( B \) nonempty subsets of a set \( S \) in the universe \( \mathcal{U} \),
\[ A - B = \emptyset \Rightarrow S - B \subseteq S - A. \]

(c) Give a \textit{proof by contradiction} of the following statement.

The \( \sqrt{2} \) is irrational.

3. Given propositions \( P \), \( Q \), \( R \) and \( S \), use your knowledge of tautologies to prove that

\[ [P \land (Q \Rightarrow R)] \Rightarrow [R \land \neg S] \]

is equivalent to

\[ [R \Rightarrow S] \Rightarrow [(\neg P \lor Q) \land (\neg P \lor \neg R)] \].
4. Give a direct proof of the following claim

\((\forall a)(\forall b)[(a, b \in \mathbb{Z} \land a > b \land a \neq b) \Rightarrow |a| = |b|].\)

5. For this problem, you may use definitions and basic properties from Calculus. Give a direct proof of the following. Over the set of real-valued functions of a real-variable,

\[(\forall f)(f \in D([3]) \Rightarrow f \in C([3])),\]

where \(D(S)\) and \(C(S)\) denote the sets of functions that are differentiable on the set \(S\) and continuous on the set \(S\), respectively.

6. Note that 3 can be written as the sum of two consecutive natural numbers; while 15 can be written as both the sum of two consecutive natural numbers and the sum of three consecutive natural numbers. Use Phase I and Phase II in pursuit of answers to the following questions.

- Can every natural number be written as the sum of at least two consecutive natural numbers? and
- If not, is there a sure fired method for knowing which ones can and which ones can’t?

Summarize your findings. With your claims, take care to distinguish your conjectures from your theorems. Your goal is to develop a list of well written statements.

7. Prove or Disprove. For each of the following statements, use Phase I and Phase II to decide on whether the given statement is true or false. If you discover that it is false, state that claim and offer a well written presentation of a counterexample (an example that illustrates that the statement is false); if true offer a well written proof.

(a) For every natural number \(n\), \(n^2 + n + 41\) is a prime.
(b) For natural numbers \(n\) and \(m\), if \(n^2 + m^2\) is even, then both \(n\) and \(m\) are even.
(c) If \(p\) and \(q\) are prime numbers such that \(p \neq q\), then \(\sqrt{p + q}\) is a prime.
Appendix B

A VIEW OF WRITING IN MATHEMATICS

In this appendix, more focused attention is given to some of what goes into presenting a proof of a proposition. A proof of a proposition is a complete justification that the proposition is true. The quality of writing (exposition) should be such that the reader can follow what is given easily.

Good exposition takes effort and time. It is important to remember that what is written should be clear, concise and self-contained. By the end of the course, any knowledgeable person reading a proof that you have written should be able to identify what is given, what you are using, the method of proof applied and what was proved.

B.1 Some Do’s and Don’t’s Regarding Exposition

Although stylistic matters can be very subjective, many of the basics of mathematical exposition are not decided according to personal preference. The following are some of the general rules of exposition that you should practice.

1. Always write in complete, well written sentences. Symbols are to be read according to the translations available and can be used only as a part of statements or equations. For example, you should not use $=$ for the word equals while you can write something like $x = y + 7$.

2. Sentences should begin with capital letters. Otherwise, capitals should be reserved for the beginning of proper nouns, sets, or other abstract objects.
3. Avoid “run on” sentences. Sentences having 35 to 40 words are almost always difficult to follow and poorly written.

4. More is not necessarily better. Don’t ramble on with mention of terms, facts or theorems that are not going to be specifically used in the proof being offered.

5. Too many connectors in a sentence is confusing and poor grammar. For example, the sentence

“If I had a hammer, then I would go to work there, since it is near here but the wages are poor so I’ll not need a hammer.”

is poor grammar for several reasons. To begin with, if, then, since, but, and so should not appear in one sentence together; the use of there and here is confusing; and the sentence is too long. A correct way to say what is intended, is

“If I had a hammer, then I would go to work at the machine factory that is near to where I live. However, since the wages are poor, I guess that I won’t need a hammer.”

6. Certain pairings of connectors are not acceptable. For example, “since” takes a comma, not a “then.” Consequently,

“Since the sky is blue, then we can go out.”

is not good grammar. A correct way to say what is intended is

“Since the sky is blue, we can go out.”

Additional rules will be discussed, as they appear to be necessary.

With consistent effort, exposition can be improved. Always read everything that you write, including symbol translations, in order to know how it reads. Some people find it particularly helpful to read what they have written out loud.

### B.1.1 Exercises

1. Reread the descriptions for the three phases of the process to proof that was described in either Chapter 2 or Appendix A.

   (a) List at least two things that you might do during Phase I: Introduction to the Problem that were not mentioned in its description.
(b) List at least two things that you might do during Phase II: Scratch Work that were not mentioned in its description.

2. For each of the following, find and formally state at least two things that you need to know (Phase I) in order to prove the given statement. Do not prove the statement.

   (a) The sum of two odd integers is an even integer.
   (b) The geometric mean of two positive real numbers does not exceed their arithmetic mean.
   (c) The sum of the measures of the interior angles for a regular convex n-gon is \( \pi \cdot (n - 2) \).
   (d) A finite Abelian group is the direct sum of cyclic groups.

3. For each of the following, describe the Don’ts that have been perpetrated.

   (a) Since when we add two consecutive primes with the smaller greater than two it is an even integer greater than two because they were odd.
   (b) But \((x + 1)\) is even since \(x\) is odd if we have that \(x < y\).
   (c) the integer \(mn\) is even and < 27.
   (d) \(B\) is a set different from \(C\) & it has more elements than \(A\).
   (e) Because an even integer is divisible by two and an odd integer is not divisible by two then the product of even integers is divisible by four.

**B.2 Trial Run**

**B.2.1 Introduction**

This section offers you an opportunity to do some self checking of your ability to carry out a proof (Phase III). You will be given an outline of the Scratch Work (Phase II) on which to build.

Your goal is to carry out the PROCESS that was described above. Consequently, your proof (Phase III) should not look like scratch work. Do the scratch work on papers other than those that you would share with anyone else. For this trial run, you can use your knowledge of plane (Euclidean) geometry and algebraic manipulations.
B.2.2 A Trial Run Problem

Prove the Pythagorean Theorem: Given a right triangle whose legs have lengths $a$ and $b$ and whose hypotenuse has length $c$, we have

$$a^2 + b^2 = c^2.$$ 

Directions:

- Use the sketch of scratch work for a proof to develop phase II towards a solution. (After working out all the details and determining the order in which to present the steps of the proof, prepare the work as if you were expecting it to be graded for mathematical correctness and quality of exposition.)

- On a clean sheet of paper, formulate a well written proof (Phase III), leaving at least a one inch margin on the right and space between lines of writing to allow room for some self grading annotations. (Remember that your proof should contain only complete sentences that are grammatically correct. It is okay to include symbols in those sentences, but they must read correctly as a part of equations or mathematical statements and should not be used to start a sentence. It should be legible and complete in itself. Be careful to specify the identity of any variables being used in the body of the proof. To make use of a diagram, present it clearly and make sure that it is well labeled.)

- Only after completing the two steps above, look at the sample write–up and use the guidelines given after our proof to do a self help check of your work.

SKETCH OF SCRATCH WORK FOR A PROOF

(A) Construct a square having sides of length $(a + b)$. Then, form a quadrilateral, with vertices on the sides of the original square, by drawing line segments joining points on neighboring sides that are respectively and alternatingly the distances $a$ and $b$ away from the vertices of the original square.

(B) Now justify that the four triangles are congruent and that the newly formed quadrilateral is a square.

(C) Finally, use the formulas for the areas of a triangle and a square to obtain two different, but equivalent, expressions for the area of the original square. Equating the expressions and simplifying should lead to the desired result.
B.2.3 A Solution for the Trial Run Problem

**Proof.** Construct a square $\square ABCD$ having sides of length $(a + b)$. Then, inscribe a quadrilateral with vertices on the sides of $\square ABCD$, by drawing line segments joining points on neighboring sides that give alternatingly the distances $a$ and $b$ away from the vertices of the original square. The constructed figure is given in the following diagram.

![Diagram](image)

By construction, we have $|HA| = |EB| = |FC| = |GD| = b$, and $|AE| = |BF| = |CG| = |DH| = a$. In addition, since $\square ABCD$ is a square,

$$m \angle HAE = m \angle EBF = m \angle FCG = m \angle GDH = 90^\circ.$$ 

Thus, by the SAS$^2$ Theorem, we conclude that $\triangle HAE \cong \triangle EBF \cong \triangle FCG \cong \triangle GDH$. Therefore, $|EF| = |FG| = |GH| = |HE| = c$ and we have four right triangles that fit the criteria for the Pythagorean Theorem.

Using corresponding angles and substitution, in addition to the facts that the sum of the measures of the angles in a triangle is $180^\circ$ and a straight angle has

---

$^2$Side Angle Side

$^3\triangle HAE \cong \triangle EBF$ means that triangle $HAE$ is congruent to triangle $EBF$. 
measure $180^\circ$, we note that

\[ m \prec EHA + m \prec HAE + m \prec AEH = 180^\circ. \]

Thus,

\[ m \prec EHA + m \prec AEH = 90^\circ \text{ and } m \prec BEF + m \prec AEH = 90^\circ. \]

Furthermore,

\[ m \prec AEH + m \prec HEF + m \prec FEB = 180^\circ \text{ yields that } m \prec HEF = 90^\circ. \]

Similarly, $m \prec EFG = m \prec FGH = m \prec GHE = 90^\circ$. Thus, \( \Box \) EFGH is a square.

Finally, comparing areas, we have

\[ (a + b)^2 = a^2 + 2ab + b^2 = 4\left(\frac{1}{2}ab\right) + c^2. \]

Simplifying leads to

\[ a^2 + 2ab + b^2 = 2ab + c^2 \]

which is equivalent to

\[ a^2 + b^2 = c^2. \]

Therefore, for a right triangle whose legs have lengths $a$ and $b$ and whose hypotenuse has length $c$, we have $a^2 + b^2 = c^2$. ■

### B.2.4 Guidelines for Self Analysis of Trial Run Problem

Now you should look at your write–up to see if you have done some things that need improvement or correction. Read each of the following: If you have missed what is advised, write the corresponding letter and number in your right side margin. In the list, m# indicates a mathematical consideration while e# indicates a matter related to exposition.
m1. A well labeled diagram can do much towards increasing clarity and conciseness. Write m1 in your right margin if you did such nonstandard things as using the lengths as labels or not naming all the important points.

m2. In the body of the proof, the square should be set forth and the construction should be described. This is because the proof needs to be complete in itself. The figure helps the reader to follow the writer’s argument. An m2 goes in the right margin if you didn’t give a careful description of your construction within your Phase III.

m3. The best you can do in the ‘construct sentence’ is to refer to the inscribed shape as a quadrilateral. That the inscribed quadrilateral was a square required proof. To prove that the inscribed quadrilateral is a square, it was necessary to justify that the four interior angles are right angles. Four sides of equal length only got you to a rhombus. If you claimed your inscribed quadrilateral was a square too quickly, m3 goes in the right margin.

m4. A common error made is in reference to angles as their measures. It is incorrect to say that “an angle plus and angle is 90°.” Since 90° is a measure of an angle, it is necessary to say “the sum of the measures of the angles is 90°.” Write m4 in your right margin if you didn’t refer to 90° etc. as measures of angles.

e5. Only use acceptable math symbols, as appropriate. For example, you should not use a “Δ” for the word triangle, a “<” for the word angle or “≅” for the word congruent standing alone in sentences. On the other hand, those symbols are acceptable as modifiers. Thus, it would be okay to refer to ΔABC, ∠D, or ΔABC ≅ ΔDEF in the body of a proof. In general, if a sentence has no formulas in it, it is best to keep everything written out in words. Write e5 in the right margin each time that you misused a symbol in either way.

e6. An effort should be made to be precise. Writing “since...since...because...so” etc. in a sentence is both verbose and cumbersome. Go over your write–up and circle each occurrence of “since,” put a rectangle around each occurrence of “thus”, and a triangle around each “so”. If you have three of any one of them, write e6 in the right margin.

e7. Write e7 in the right margin for each occurrence of violations of either of the following: (a) Avoid starting more than two consecutive sentences with the same lead in term such as: thus, hence, therefore, etc. (b) Terms like “thus”
and “hence” are intended to be followed by something that is concluded from work that has gone before: They should not be used to lead into a string of simplifications. (Expressions like “it follows that” or “simplifying” serve nicely for the latter purpose.)

e8. You need not repeat or state definitions as justifications. You can simply make use of them as you translate parts of your set–up for a proof to forms that you will be using in your logical arguments. Write e8 across any instances of repeating things where it was unnecessary.

e9. Finally, check your sentence starts. If the first line after a period ending a sentence is not started with a capitalized word, e9 goes in the right margin.

**B.2.5 Exercises**

1. Give your proof for the Trial Run Problem another careful reading, remembering to read everything that you wrote. Based on this reading, spend a few moments trying to make improvements.

2. Analyze your performance on the Trial Run Problem. Include at least three things that you learned from the experience.

3. Left to their own devices, some students try to solve the Trial Run Problem by applying the Law of Cosines. Find the sound mathematical reason why this is not an acceptable approach and briefly describe it.

4. Briefly describe at least three personal goals for your own approach to doing mathematics.