MATH 245 FINAL, WINTER 2025

Rules. The final is open book and open notes. But you are not allowed to use any other resources such as the internet or classmates.

Due Time. Thursday March 20th by noon.

Submission Method. You could either email me a pdf copy of your solutions or drop by my office MSB 3220 between 10am and noon.

1. Let $f(n)$ denote the number of fixed-point free involutions $\omega \in \mathfrak{S}_{2n}$ (i.e., $\omega^2 = 1$, and $\omega(i) \neq i$ for all $i \in [2n]$). Find a simple expression for $\sum_{n \geq 0} f(n) x^n / n!$. (Set $f(0) = 1$.)

2. **(a)** Let E_n denote the poset of all subsets of $[n]$ whose elements have even sum, ordered by inclusion. Find $\#E_n$.

(b) Compute $\mu(S, T)$ for all $S \leq T$ in E_n.

3. For a finite lattice L, let $f_L(m)$ be the number of m-tuples $(t_1, \ldots, t_m) \in L^m$ such that $t_1 \wedge t_2 \wedge \cdots \wedge t_m = \hat{0}$. Give two proofs that

$$f_L(m) = \sum_{t \in L} \mu(\hat{0}, t)(\# \bigvee\nolimits_t)^m.$$

The first proof should be by direct Möbius inversion, and the second by considering $(\sum_{t \in L} t)^m$ in the Möbius algebra $A(L, \mathbb{R})$.

4. Given a finite poset $\mathcal{P}$, the *order polytope* of $\mathcal{P}$, denoted by $\mathcal{O}(\mathcal{P})$, is the collection of functions $\boldsymbol{x} \in \mathbb{R}^{\mathcal{P}}$ satisfying

- $0 \leq x_a \leq 1$, for all $a \in \mathcal{P}$, and
- $x_a \leq x_b$, if $a \leq b$ in $\mathcal{P}$.

(Equivalently, $\mathcal{O}(\mathcal{P})$ can be defined as the convex hull of incidence vectors of dual order ideals of $\mathcal{P}$. See Example 4.6.17. Hence, $\mathcal{O}(\mathcal{P})$ is an integral polytope.)

Let U_k be the ordinal sum of k copies of 2-element antichains. Compute the generating function

$$\sum_{n \geq 0} i(\mathcal{O}(U_k), n) x^n$$

of the Ehrhart polynomial of the order polytope $\mathcal{O}(U_k)$ of U_k. Your answer should be a rational function reduced to lowest terms.