Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #88, 12 pp.
Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

On the Todd Class of the Permutohedral variety

Federico Castillo[*1] and Fu Liu[†2]

[1] *Department of Mathematics, University of Kansas, KS, USA*
[2] *Department of Mathematics, University of California, Davis, CA, USA*

Abstract. In the special case of braid fans, we give a combinatorial formula for the Berline–Vergne's construction for an Euler-Maclaurin type formula that computes number of lattice points in polytopes. Our formula is obtained by computing a symmetric expression for the Todd class of the permutohedral variety. By showing that this formula does not always have positive values, we prove that the Todd class of the permutohedral variety X_d is not effective for $d \geq 24$. Additionally, we prove that the linear coefficient in the Ehrhart polynomial of any lattice generalized permutohedron is positive.

Keywords: Ehrhart polynomials, generalized permutohedra, Berline–Vergne construction

1 Introduction

Let Λ be a lattice of finite rank and $V = \Lambda \otimes \mathbb{R}$ be the corresponding real finite-dimensional vector space. A **lattice polyope** in V is a polytope such that all of its vertices lie in Λ. A classical problem in the crossroads between enumerative combinatorics and discrete geometry is that of counting lattice points in lattice polytopes. For any polytope $P \subset V$ we define $\text{Lat}(P) := |P \cap \Lambda|$. One of the earliest results in the area is Pick's theorem, which says that for any lattice polygon $P \subset \mathbb{R}^2$ we have

$$\text{Lat}(P) = a(P) + \frac{1}{2}b(P) + 1,$$

where $a(P)$ is the area of P and $b(P)$ is the number of lattice points on the boundary of P. One way to obtain a higher dimensional analog of Pick's formula is to find a formula relating the number of lattice points of P with the different normalized volumes of the faces F of P. We want a real-valued function α on pairs (F, P), where F is a face of a lattice polytope P, such that

$$\text{Lat}(P) = \sum_{F\text{: a face of } P} \alpha(F, P)\, \text{nvol}(F), \tag{1.1}$$

[*] fcastillo@ku.edu. Partially supported by NSF grant DMS–1440140
[†] fuliu@math.ucdavis.edu Partially supported by the Simons Foundation grant #426756 and by NSF grant DMS–1440140

where $\mathrm{nvol}(F)$ is the normalized volume of F. It is clear that for a given lattice polytope P one can always find many functions α satisfying (1.1). What we want is a function that works simultaneously for *all* lattice polyopes. We can do this by requiring the function α to be *local*, i.e., if the numbers $\alpha(F, P)$ only depend on the local geometry of P around F, or more specifically, the value only depends on $\mathrm{ncone}(F, P)$, the normal cone of P at F. Any local function α that satisfies (1.1) for all lattice polytopes P is called a **McMullen function**, since McMullen was the first to prove their existence [9]. His proof is nonconstructive and shows that there are infinitely many McMullen functions. In the present paper we compute the values for a particular McMullen function on a special family of polytopes: generalized permutohedra. A **generalized permutohedron** is a polytope whose normal fan is a coarsening of the braid fan Σ_d.

Our methods for computing a McMullen function for generalized permutohedra are based on the theory of toric varieties.

1.1 Todd classes of toric varieties

Let P be a lattice polytope with normal fan Σ and X_Σ be the associated toric variety. The **Todd class** $\mathrm{Td}(X_\Sigma)$ is an element in the Chow ring of X_Σ. As such it can be written as a $\mathbb{Q}$-linear combination of the toric invariant cycles $[V(\sigma)]$:

$$\mathrm{Td}(X_\Sigma) = \sum_{\sigma \in \Sigma} r_\Sigma(\sigma)\ [V(\sigma)], \quad r_\Sigma(\sigma) \in \mathbb{Q}. \tag{1.2}$$

Since the cycles $[V(\sigma)]$ satisfy algebraic relations, the values $r_\Sigma(\sigma)$ satisfying (1.2) are not uniquely determined. An amazing connection with lattice polytopes is given by the fact that any function $r_\Sigma(\cdot)$ satisfying (1.2) defines a function α satisfying (1.1) for P by setting

$$\alpha(F, P) = r_\Sigma(\mathrm{ncone}(F, P)).$$

A proof of this fact can be found in Danilov's 1978 survey [4] where he further asked if there exist a function r that depends only on the cone σ and not on Σ, in other words, if there exist a *local* function r satisfying (1.2) for all fans Σ. Accordingly, we call such a function r on pointed cones a *Danilov function*. By setting

$$\alpha(F, P) = r(\mathrm{ncone}(F, P)),$$

any Danilov function gives a McMullen function.

We want to briefly remark on two constructions of Danilov functions from the last two decades. Pommersheim and Thomas [10] gave a construction of a Danilov function $r(\sigma)$ that depends on choosing a *complement map* for subspaces. A couple of years later Berline and Vergne [1] constructed a McMullen function with the property that it is computable in polynomial time fixing the dimension and it is a valuation on cones. We call this construction the BV-function, and denote it by α^{BV}.

Later they showed that if a function r on pointed cones is defined by $r(\sigma) = \alpha^{\text{BV}}(F, P)$ as long as $\sigma = \text{ncone}(F, P)$, then it is a Danilov function. For convenience, we abuse the notation, and consider α^{BV} to be both a function on pairs (F, P) and a function on cones with the connection that $\alpha^{\text{BV}}(F, P) = \alpha^{\text{BV}}(\text{ncone}(F, P))$. Thus, α^{BV} is both a McMullen function and a Danilov function.

Both constructions, Berline–Vergne's and Pommersheim-Thomas', are algorithmic. A priori it is very hard to get formulas for general cones. There are very few examples of fans Σ for which $\alpha^{\text{BV}}(\sigma)$ (or any other Danilov function) have been computed for all $\sigma \in \Sigma$. In this paper, we focus on computing the BV-function on all cones in braid fans using tools developed in previous work of the authors.

In [2] we exploited an extra symmetry property satisfied by the function α^{BV}, and used this symmetry to study the values on cones in **braid fans**. One main result in [2] is the *uniqueness theorem*, which in the context of the present paper states that, for the specific example of braid fans, α^{BV} is the unique function satisfying (1.2) and being invariant under the permutation action of the symmetric group on the ambient space. Using this, we obtain the main result of this paper – Theorem 4.1 – which gives a combinatorial formula for α^{BV} on all cones in braid fans.

1.2 Connection to Ehrhart theory

Ehrhart proved that for every lattice polytope P the function $\text{Lat}(tP)$ for $t \in \mathbb{N}$ is a polynomial in t of dimension $d = \dim P$. More precisely, there exist $a_0, a_1, \ldots, a_d \in \mathbb{Q}$ such that for all $t \in \mathbb{N}, \text{Lat}(tP) := a_0 + a_1 t^1 + a_2 t^2 + \cdots + a_d t^d$. The right hand side is called the **Ehrhart polynomia**l of P. Given a McMullen formula α one can deduce that

$$a_k = \sum_{\substack{F:\text{a face of } P \\ \dim F = k}} \alpha(F, P) \,\text{nvol}(F). \tag{1.3}$$

We call a lattice polytope P **Ehrhart positive** if all the (middle) coefficients of its Ehrhart polynomial are positive (see [8] for a recent survey on Ehrhart positivity). One of the main motivations for [2] was to prove a conjecture of De Loera et al. asserting that matriod polytopes are Ehrhart positive [5]. Noticing that matroid polytopes belong to the family of generalized permutohedra, we focus on the latter larger family of polytopes.

Conjecture 1.1 ([2, Conjecture 1.2])**.** *Lattice generalized permutohedra are Ehrhart positive.*

One observes that a consequence of (1.3) is that if we have a McMullen function α such that $\alpha(F, P)$ is positive for all faces $F \subset P$ then P is Ehrhart positive. (The converse is not true as shown in Section 3.4 of [3].) Using the fact that the BV-function α^{BV} is a McMullen function and it has certain valuation properties, we showed in [2] that the following conjecture (if true) implies Conjeture 1.1.

Conjecture 1.2 ([2, Conjecture 1.3]). *Let P be a generalized permutohedron and $F \subset P$ a face, then $\alpha^{\mathrm{BV}}(F, P)$ is positive. Equivalently, $\alpha^{\mathrm{BV}}(\sigma)$ is positive for every cone σ in the braid fan.*

Despite of these positive results we've obtained in our previous work towards Conjeture 1.2, in the present paper, we use our main result - the combinatorial formula described in Theorem 4.1 - to find negative values for α^{BV} on some cones in braid fans, hence disproving Conjeture 1.2. Note that this does not imply that Conjeture 1.1 is false, and in fact we present a proof, independent of the rest of the paper, that the linear coefficient of the Ehrhart polynomial of any lattice generalized permutohedron is positive, providing further evidence to Conjeture 1.1. This positivity result of linear Ehrhart coefficient was proved independently by Jochemko and Ravichandran in [7], using different techniques from what are presented in this paper.

2 Preliminaries and notation.

Here we review concepts and notation that we are going to use. As standard we denote $[d+1] := \{1, 2, 3, \cdots, d, d+1\}$. The set of all subsets of $[d+1]$ form a poset $\mathcal{B}_{d+1}$ called the **boolean algebra** and we define the **truncated boolean algebra**, denoted by $\overline{\mathcal{B}}_{d+1}$, to be the poset obtained from $\mathcal{B}_{d+1}$ by removing $[d+1]$ and $\emptyset$. Two elements $S, S' \in \overline{\mathcal{B}}_{d+1}$ are **incomparable** if neither $S \subseteq S'$ nor $S \supseteq S'$. A k**-chain** $S_\bullet = (S_1, \cdots, S_k)$ is a sequence of k totally ordered elements of $\overline{\mathcal{B}}_{d+1}$. For notational purposes, we complete $S_\bullet$ by adding $\emptyset$ and $[d+1]$ to obtain $\emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq S_{k+1} = [d+1]$. We denote the new sequence by $\widehat{S_\bullet}$. The set of all k-chains in $\overline{\mathcal{B}}_{d+1}$ is denoted $\mathcal{C}^k_{d+1}$ and let $\mathcal{C}_{d+1} = \bigcup_k \mathcal{C}^k_{d+1}$.

2.1 Braid fan and Permutohedral variety

Let V_d be the d-dimensional real vector space $\mathbf{1}^\perp \subset \mathbb{R}^{d+1}$, where $\mathbf{1}$ is the all one vector. Its dual is $W_d = \mathbb{R}^{[d+1]}/(\mathbf{1})$.

The combinatorics of the braid fans are summarized in Lemma 2.1 below. Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{d+1}$ be the standard basis of $\mathbb{R}^{d+1}$ and for each $S \in \overline{\mathcal{B}}_{d+1}$ we define $\boldsymbol{e}_S := \sum_{i \in S} \boldsymbol{e}_i$ as an element in W_d. For any k-chain $S_\bullet$ of $\overline{\mathcal{B}}_{d+1}$, we define the corresponding **braid cone** $\sigma_{S_\bullet} := \mathrm{Cone}(\boldsymbol{e}_S : S \in S_\bullet)$, which is k-dimensional. The following is well known.

Lemma 2.1. *The map $S_\bullet \mapsto \sigma_{S_\bullet}$ gives a one-to-one correspondence between chains in $\mathcal{C}_{d+1}$ and cones in the braid fan Σ_d. Moreover, k-chains in $\mathcal{C}_{d+1}$ are in bijection with k-dimensional cones in Σ_d.*

2.2 Permutohedral variety

For toric varieties we follow the notation and terminology of [6]. The **permutohedral variety** X_d is the toric variety associated to Σ_d. For each $S_\bullet \in \mathcal{C}_{d+1}$, its corresponding braid cone $\sigma_{S_\bullet}$ is associated with a subvariety $V(\sigma_{S_\bullet})$. These subvarieties are the **torus invariant cycles**.

For any $d \in \mathbb{N}$ we define the following ring

$$R_d := \mathrm{k}[x_S : S \in \overline{\mathcal{B}}_{d+1}],$$

where k is any algebraically closed field. For any element $i \in [d+1]$ we define the linear form $\ell_i := \sum_{S \ni i} x_S$.

Definition 2.2. The **Chow ring of the permutohedral variety** X_d can be presented as $A_d \cong R_d/(I_1 + I_2)$, where

$$I_1 = \langle x_S x_{S'} : \text{ for } S, S' \text{ incomparable}\rangle, \qquad I_2 = \langle \ell_a - \ell_b : \text{ for all } a, b \in [d+1]\rangle.$$

We are interested in computing the Todd class of X_d in A_d. The following definition follows [6, Section 5].

Definition 2.3. The **Todd class of** X_d is the element of A_d defined as

$$\mathrm{Td}(X_d) := \prod_{S \in \overline{\mathcal{B}}_{d+1}} \left(\frac{x_S}{1 - e^{-x_S}} \right), \tag{2.1}$$

which is an element of A_d by expanding each parenthesis on the right hand side as

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x^{2i} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \cdots . \tag{2.2}$$

Here B_i is the i-th Bernoulli number. Also note that $f^k = 0$ for any $k > d$ and $f \in A_d$, so the sum in (2.2) is finite.

For each $S_\bullet \in \mathcal{C}_{d+1}$, the class of the subvariety $V(\sigma_{S_\bullet})$ in A_d is denoted $[V(\sigma_{S_\bullet})]$, and it can be represented as a square-free element in A_d :

$$[V(\sigma_{S_\bullet})] = x_{S_\bullet} := \prod_{S \in S_\bullet} x_S. \tag{2.3}$$

We are interested in expressions for $\mathrm{Td}(X_d)$ in terms of classes of the torus invariant cycles. In other words, we are looking for $r(S_\bullet) \in \mathbb{Q}$ such that

$$\mathrm{Td}(X_d) = \sum_{S_\bullet \in \mathcal{C}_{d+1}} r_d(S_\bullet)\, x_{S_\bullet} = \sum_{S_\bullet \in \mathcal{C}_{d+1}} r_d(S_\bullet)\, [V(\sigma_{S_\bullet})]. \tag{2.4}$$

We call such an expression a **square-free expression for the Todd class** $\mathrm{Td}(X_d)$ **of** X_d. Our interest in such an expression lies in the following theorem originally attributed to Danilov which is already mentioned in Section 1.1. Here we only state it in the particular case of braid fans.

Theorem 2.4 (Section 5 in [6])**.** *Let P be a d-dimensional lattice generalized permutohedron with normal fan Σ_d. Suppose r_d is a function defined on $\mathcal{C}_{d+1}$ such that (2.4) holds. Using the one-to-one correspondence between chains in $\mathcal{C}_{d+1}$ and cones in Σ_d described in Lemma 2.1, we can consider r_d to be a function on braid cones by letting $r_d(\sigma_{S_\bullet}) := r_d(S_\bullet)$. Then we have that*

$$\mathrm{Lat}(P) = \sum_{F \subset P} r_d(\mathrm{ncone}(F, P)) \, \mathrm{nvol}(F). \tag{2.5}$$

Therefore, an equation of the form (2.4) gives a solution to (1.1) for lattice generalized permutohedra by setting $\alpha(F, P) = r_d(\mathrm{ncone}(F, P))$.

We are going to require one more special property for our expressions of the form (2.4).

Definition 2.5. The symmetric group $\mathfrak{S}_{d+1}$ acts on elements of $\overline{\mathcal{B}}_{d+1}$ hence on the generators of the ring R_d. Notice that this action fixes both ideals I_1 and I_2 so that $\mathfrak{S}_{d+1}$ acts naturally on A_d too. We say an element $f \in A_d$ is **symmetric** if $\pi \cdot f = f$ for all $\pi \in \mathfrak{S}_{d+1}$.

For any $f \in A_d$, we define its **symmetrization** to be

$$f^\sharp := \frac{1}{(d+1)!} \sum_{\pi \in \mathfrak{S}_{d+1}} \pi \cdot f. \tag{2.6}$$

(It is easy to see that $f^\sharp$ is symmetric.)

Recall that the BV-function α^{BV} is both a McMullen function and a Danilov function. In the case of the braid fan, we abuse notation again, and consider α^{BV} a function on $\mathcal{C}_{d+1}$ by letting $\alpha^{\mathrm{BV}}(S_\bullet) := \alpha^{\mathrm{BV}}(\sigma_{S_\bullet}), \quad \forall S_\bullet \in \mathcal{C}_{d+1}$. Then using results from [2] we prove the following.

Theorem 2.6. *[Theorem 5.5 in [2]] There is a unique symmetric square-free expression for* $\mathrm{Td}(X_d)$*. It is given by the Berline–Vergne function:*

$$\mathrm{Td}(X_d) = \sum_{S_\bullet \in \mathcal{C}_{d+1}} \alpha^{\mathrm{BV}}(S_\bullet)[V(\sigma_{S_\bullet})]. \tag{2.7}$$

We call the right hand side of (2.7) the **Berline–Vergne expression** for the Todd class of X_d.

Combining the theorem with the symmetrization described in (2.6) we get the following

Proposition 2.7. *Let f be any square-free expression for* $\mathrm{Td}(X_d)$ *(as in (2.4)), then its symmetrization $f^\sharp$ is the Berline–Vergne expression for* $\mathrm{Td}(X_d)$*.*

3 Spider diagrams

In this section we develop the necessary combinatorial language that will be used to express our main formulas in Section 4.

Definition 3.1. Let $T_\bullet \in \mathcal{C}_{d+1}$ and $S \in T_\bullet$ (so S is a subset of $[d+1]$). A **spider** $\mathrm{Sp} = \mathrm{Sp}(T_\bullet, S)$ on $T_\bullet$ with head S is a graph on the vertex set $T_\bullet$ with edge set $\{S, T\}$ for every $T \in T_\bullet \setminus \{S\}$. We call S the **head** and every non head vertex a **leg**. Legs are partitioned in two subsets L and R. The set L consists of the **left legs**, i.e., of $T \in \mathcal{T}_\bullet$ such that $T \subset S$ and R consists of the **right legs**, sets $T \in \mathcal{T}_\bullet$ with $S \subset T$.

The **size** of a spider is $|\mathrm{Sp}(T_\bullet, S)| := |T_\bullet|$, the size of its vertex set. A spider of size one is called a **trivial** spider. It has no legs. (Note that number of edges in a spider Sp is $|\mathrm{Sp}| - 1$.)

Notation 3.2. A left leg will be labeled as T_i^L if it is the i-th *smallest* vertex among all left legs, and a right leg will be labeled by T_j^R if it is the j-th *largest* vertex among all right legs. If there are no left legs, we give the head vertex h an additional label T_1^L; similarly, if there are not Right legs, we give the head vertex S an additional label T_1^R.

Example 3.3. To save space we avoid commas, for instance $\{12\} := \{1,2\}$. Consider the spider with chain $T_\bullet = \{12\} \subset \{123\} \subset \{123456\} \subset \{12345678\}$ and head $S = \{12\}$. In Figure 1 we have labeled the spider according to Notation 3.2. Note that the head S receives also the label of T_1^L.

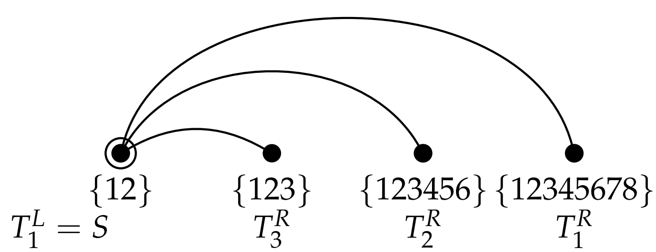

Figure 1: A spider with the head having two different labels.

Definition 3.4. Let $T_\bullet \in \mathcal{C}_{d+1}$ be a chain. A **spider diagram** D consist of a partition of $T_\bullet$ into k-disjoint intervals $T_{1,\bullet}, \cdots, T_{k,\bullet}$ together with a spider $\mathrm{Sp}_i := \mathrm{Sp}(T_{i,\bullet}, S_i)$ on each interval. Notice that the set of heads form a chain $S_\bullet \in \mathcal{C}_{d+1}$. Additionally, we always adjoin two trivial spiders Sp_0 and Sp_{k+1} having vertex set $\emptyset$ and $[d+1]$ respectively.

Notation 3.5. In a spider diagram D the legs are now triply indexed: the element $T_{i,j}^P$ with $P \in \{L, R\}$ is the jth smallest/largest on the side L/R of the i-th spider. See Figure 2 for an example of this indexing system.

Example 3.6. In Figure 2 we show a spider diagram with two spiders with vertices labeled according to Notation 3.5. Notice the two trivial spiders on the extremes.

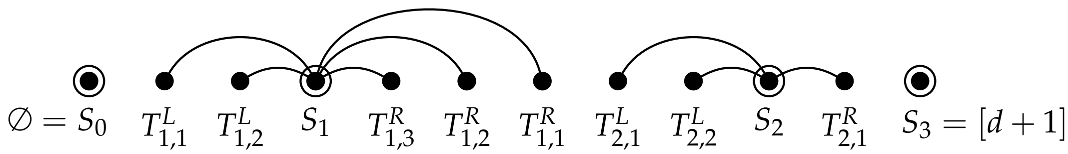

Figure 2: A spider diagram with two spiders.

For our formulas in the next section we to define several terms coming from spider diagrams.

Definition 3.7. Let $T_\bullet \in \mathcal{C}_{d+1}$ be a chain and let D be a spider diagram on it. We denote the spiders $\mathrm{Sp}_i = (G_i, T_{i,\bullet}, S_i)$ for $i = 0, \cdots, k+1$, where Sp_0 and Sp_{k+1} are trivial. For each i, let m_i its size, and L_i and R_i be set of left and right legs respectively.

We define the **internal weight** of a single spider Sp_i as

$$\mathrm{intwt}(\mathrm{Sp}_i) := \left(\prod_{j>1} \frac{|T^L_{i,j} - T^L_{i,j-1}|}{|S_i - T^L_{i,j-1}|} \right) \left(\prod_{j>1} \frac{|T^R_{i,j-1} - T^R_{i,j}|}{|T^R_{i,j-1} - S_i|} \right), \tag{3.1}$$

and the **boundary weight** of the diagram D as

$$\mathrm{bdwt}(D) := \prod_{i=1}^{k+1} \left(\frac{|T^L_{i,1} - T^R_{i-1,1}|}{|S_i - S_{i-1}|} \right). \tag{3.2}$$

Notice that the internal weights of the extremal spiders is 1.

The **weight** of a spider diagram D is defined as

$$\mathrm{wt}(D) := \mathrm{bdwt}(D) \prod_{i=0}^{k+1} \mathrm{intwt}(\mathrm{Sp}_i).$$

The **sign** of D is defined as $\mathrm{sgn}(D) := (-1)^{|D|-k}$. The **Tcoefficient** of D is the coefficient of the monomial $\prod x_{S_i}^{m_i}$ in the expression for $\mathrm{Td}(X_d)$ given by expanding (2.1) using (2.2) on each parenthesis. The **binomial** of D is given by $\mathrm{Binom}(D) := \prod \binom{|\mathrm{Sp}_i|-1}{|L_i|,|R_i|}$.

Remark 3.8. The extremal spiders only affect the weight. In general, when we refer to the heads of a spider diagram we ignore $\emptyset$ and $[d+1]$.

4 General formula

Theorem 4.1. *We have the following square-free expression*

$$\mathrm{Td}(X_d) = \sum_{S_\bullet \in \mathcal{C}_{d+1}} \alpha^{\mathrm{BV}}(S_\bullet)[V(\sigma_{S_\bullet})]. \tag{4.1}$$

where

$$\alpha^{\mathrm{BV}}(S_\bullet) = \sum_{D \in \mathcal{SD}(\mathcal{S}_\bullet)} \mathrm{Tdcoeff}(D)\mathrm{Binom}(D)\mathrm{sgn}(D)\mathrm{wt}(D), \tag{4.2}$$

where $\mathcal{SD}(\mathcal{S}_\bullet)$ is the set of all spider diagrams with heads $S_\bullet$.

The proof comes from directly computing using the definition of Todd class in (2.1) and using the relations in I_2 to express arbitrary monomials as a sum of *square-free* monomials. Every time we make a choice we average over all possible choices to keep symmetry which allow us to invoke Theorem 2.6.

Proposition 4.2. *The number of terms in (4.2) has the following generating function*

$$\sum_{n=1}^{\infty} h(n)z^n = -\frac{z(z^4 - 2z^2 + 2z + 1)}{z^5 - z^4 - 2z^3 + 4z^2 + z - 1} = z + 3z^2 + 5z^3 + 15z^4 + 29z^5 + \cdots . \tag{4.3}$$

4.1 Formulas for Berline–Vergne function

To see how the (4.2) works we are going to compute some examples.

Proposition 4.3 (Codimension 2 cones.)**.** *Let $(S_1, S_2) \in \mathcal{C}_{d+1}$ be an arbitrary 2-chain, then*

$$\alpha^{\mathrm{BV}}(S_1, S_2) = \frac{1}{4} - \frac{1}{12}\left(\frac{d+1-s_2}{d+1-s_1} + \frac{s_1}{s_2}\right), \tag{4.4}$$

where $s_i := |S_i|$ for $i = 1, 2$.

Proof. We use Theorem 4.1. In this case all possible spider diagrams are shown in Figure 3. □

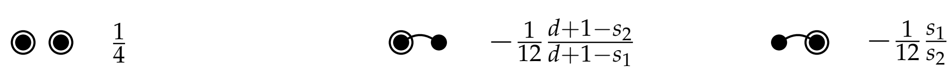

Figure 3: All spider diagrams on two vertices with the corresponding contribution to (4.2). Extremal spiders are omitted from the figure.

Formula (4.4) (and a similar one for three dimesional cones) was already obtained in [2] relying on some general formulas in the Berline–Vergne constructions. Since there is no simple closed formula for their construction for unimodular cones of dimension larger than three, we couldn't push it further than that. The next proposition shows Theorem 4.1 in action. This formula couldn't be obtained with previously known tools.

Proposition 4.4. *Let $(S_1, S_2, S_3, S_4) \in \mathcal{C}_{d+1}$ be an arbitrary 4-chain, then*

$$\begin{aligned}
\alpha^{\mathrm{BV}}(S_1, S_2, S_3, S_4) = \frac{1}{16} - \frac{1}{48}\left(\frac{s_3 - s_2}{s_3 - s_1} + \frac{s_1}{s_2} + \frac{s_4 - s_3}{s_4 - s_2} + \frac{s_2 - s_1}{s_3 - s_1} + \frac{d+1-s_4}{d+1-s_3} + \frac{s_3 - s_2}{s_4 - s_2}\right) \\
+ \frac{1}{144}\left(\frac{s_3 - s_2}{s_3 - s_1}\frac{d+1-s_4}{d+1-s_3} + \frac{s_3 - s_2}{s_4 - s_1} + \frac{s_1}{s_2}\frac{d+1-s_4}{d+1-s_3} + \frac{s_1}{s_2}\frac{s_3 - s_2}{s_4 - s_2}\right) \\
+ \frac{1}{720}\left(\frac{s_3 - s_2}{s_3 - s_1}\frac{s_4 - s_3}{s_4 - s_1}\frac{d+1-s_4}{d+1-s_1} + 3\frac{s_1}{s_2}\frac{s_4 - s_3}{s_4 - s_2}\frac{d+1-s_4}{d+1-s_2}\right) \\
+ \frac{1}{720}\left(3\frac{s_1}{s_2}\frac{s_2 - s_1}{s_3 - s_1}\frac{d+1-s_4}{d+1-s_3} + \frac{s_1}{s_2}\frac{s_2 - s_1}{s_4 - s_2}\frac{s_3 - s_2}{s_4 - s_2}\right)
\end{aligned}$$

where $s_i := |S_i|$ for $i = 1, \cdots, 4$.

Proof. We use Theorem 4.1. Figure 4 shows all possible spider diagrams in this case. □

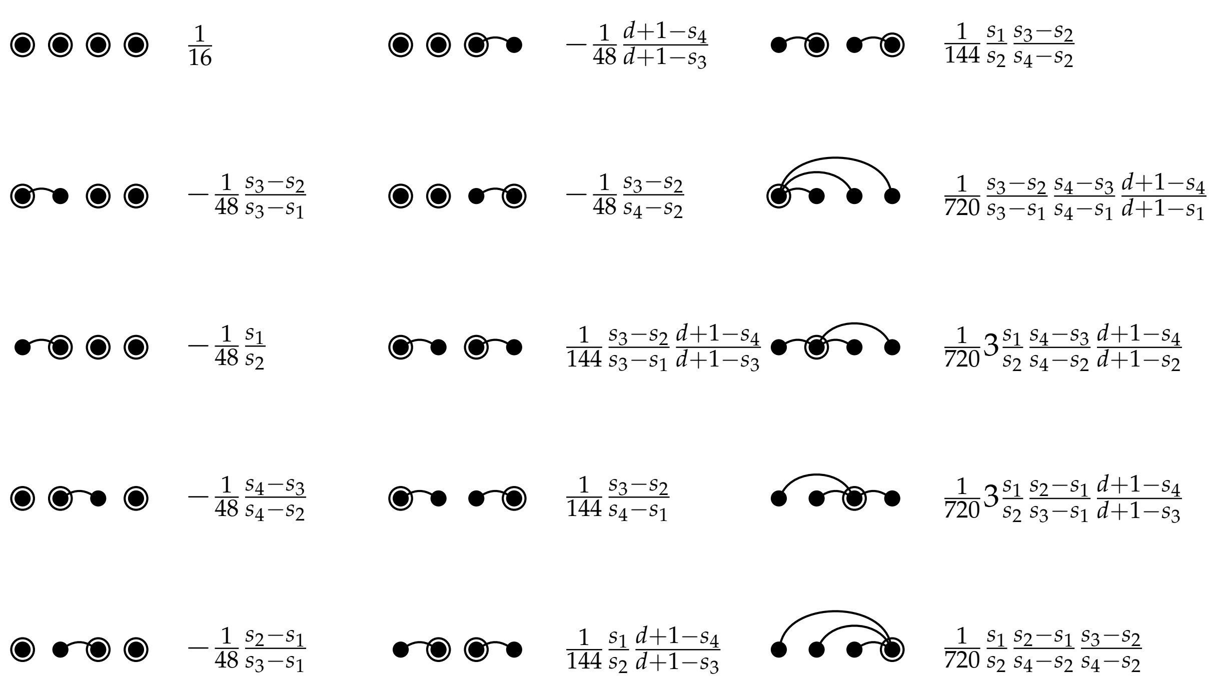

Figure 4: All spider diagrams on four vertices with the corresponding contribution to Equation (4.2). Extremal spiders are omitted from the figure.

Example 4.5. Using SAGE we found negative values for $\alpha_d$ffl in four dimensional cones in Σ_d. The smallest d for which this happens is $d+1 = 25$, where $\alpha_{24}(S_1, S_2, S_3, S_4) = -19/1684800$, for any four chain with $|S_1| = 10, |S_2| = 12, |S_3| = 13, |S_4| = 15$.

Example 4.5 disproves Conjeture 1.2. Furthermore, it also enable us to prove the following theorem.

Theorem 4.6. *The Todd class of the permutohedral variety X_d is not effective for $d \geq 24$. That is, there is no way of expressing it as a nonnegative combination of the cycles.*

5 Edge positivity

As mentioned in the introduction for every lattice polytope P the function $\mathrm{Lat}(tP), t \in \mathbb{N}$ is a polynomial in t of dimension $d = \dim P$, i.e., $\mathrm{Lat}(tP) = a_0 + a_1 t^1 + a_2 t^2 + \cdots + a_d t^d, \quad a_i \in \mathbb{Q}$. This is the **Ehrhart polynomial** of P and will be denoted $\mathrm{Lat}(P, t)$. We also define $\mathrm{Lat}^i(P) := [t^i]\mathrm{Lat}(P, t)$, the coefficient of t^i in the Ehrhart polynomial.

5.1 Edge positivity

In this section we take a different argument to show that α^{BV} values are indeed positive on codimension one cones in the braid fan and thus the main conjecture Conjeture 1.1 is true for Lat^1. The arguments in this section are independent of the rest of the paper. We make use of hypersimplices.

Proposition 5.1. *If $\mathrm{Lat}^1(\Delta_{k,d+1}) > 0$ for all $1 \leq k \leq n$ then α^{BV} is positive on every codimension one cone, thus $\mathrm{Lat}^1(P) > 0$ for any generalized permutohedra.*

Proof. This is a consequence of [2, Theorem 5.5]. In the case of an edge the mixed valuation is equal to the valuation itself, the rest of the formula is positive hence the first part follows. The second part is a consequence of the reduction theorem [2, Theorem 3.5] which shows how the positivity of α^{BV} for all codimension k cones in Σ_d implies positivity of Lat^k for all generalized permutohedra. □

The following result is standard [11, Chapter 3, Ex. 62].

Proposition 5.2. *The Ehrhart polynomial for $\Delta_{k,d+1}$ is given by*

$$\mathrm{Lat}(\Delta_{k,d+1}, t) = [z^{kt}] \left(\frac{1 - z^{t+1}}{1 - z} \right)^{d+1}. \tag{5.1}$$

Or equivalently the more explicit formula

$$\mathrm{Lat}(\Delta_{k,d+1}, t) = \sum_{i=0}^{k} (-1)^i \binom{d+1}{i} \binom{d + t(k-i) - i}{d} \tag{5.2}$$

Lemma 5.3. *For any $k \leq d$,* $\mathrm{Lat}^1(\Delta_{k,d+1}) > 0$.

The proof is a careful tracking of the linear coefficient in (5.2).

Theorem 5.4. *Conjeture 1.1 is true for the linear terms. More precisely,* $\mathrm{Lat}^1(P) > 0$ *for every lattice generalized permutohedron P.*

Proof. It follows from Proposition 5.1 and Lemma 5.3. □

Acknowledgements

This project started in Fall 2017 when both authors were attending the program "Geometric and Topological Combinatorics" at the Mathematical Sciences Research Institute in Berkeley, California. We thank Jamie Pommersheim for many helpful discussions.

References

[1] N. Berline and M. Vergne. "Local Euler-Maclaurin formula for polytopes". *Mosc. Math. J.* **7**.3 (2007), pp. 355–386, 573. Link.

[2] F. Castillo and F. Liu. "Berline-Vergne valuation and generalized permutohedra". *Discrete Comput. Geom.* **60**.4 (2018), pp. 885–908. Link.

[3] F. Castillo, F. Liu, B. Nill, and A. Paffenholz. "Smooth polytopes with negative Ehrhart coefficients". *J. Combin. Theory Ser. A* **160** (2018), pp. 316–331. Link.

[4] V. Danilov. "The geometry of toric varieties". *Russian Math. Surveys* **33**.2 (1978), pp. 97–154.

[5] J. A. De Loera, D. C. Haws, and M. Köppe. "Ehrhart polynomials of matroid polytopes and polymatroids". *Discrete Comput. Geom.* **42**.4 (2009), pp. 670–702. Link.

[6] W. Fulton. *Introduction to toric varieties*. Vol. 131. Annals of Mathematics Studies. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993, pp. xii+157.

[7] K. Jochemko and M. Ravichandran. "Generalized permutahedra: Minkowski linear functionals and Ehrhart positivity". 2019. arXiv:1909.08448.

[8] F. Liu. "On positivity of Ehrhart polynomials". *Recent trends in algebraic combinatorics*. Vol. 16. Assoc. Women Math. Ser. Springer, Cham, 2019, pp. 189–237.

[9] P. McMullen. "Valuations and dissections". *Handbook of convex geometry, Vol. A, B*. North-Holland, Amsterdam, 1993, pp. 933–988.

[10] J. Pommersheim and H. Thomas. "Cycles representing the Todd class of a toric variety". *J. Amer. Math. Soc.* **17**.4 (2004), pp. 983–994. Link.

[11] R. P. Stanley. *Enumerative combinatorics. Volume 1*. Second. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012, pp. xiv+626.