On bijections between rooted trees and the comb basis for the cohomology of the weighted partition poset

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Outline

- Introduction
  - Poset homology and cohomology
  - Known results on weighted partition poset
- Questions and results
  - González D'león-Wachs' question
  - Discussion and rephrasing
  - Conjectures and results

Basic definitions

Given a poset (partially ordered set) P, a *chain* of P is a totally ordered subset of P, and a *j-chain* is chain of $j+1$ elements.

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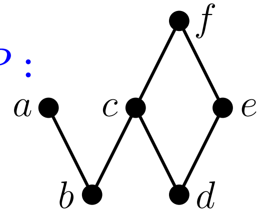
Given a poset P, the *order complex* of P, denoted by $\Delta(P)$ is an (abstract) simplicial complex, where the vertices are the elements of P and the faces are the chains of P.

$P:$



$$\Delta(P) = \left\{ \begin{array}{c} \emptyset, a, b, c, d, e, f, \\ ab, bc, bf, cd, cf, de, df, ef, \\ bcf, cdf, def \end{array} \right\}$$

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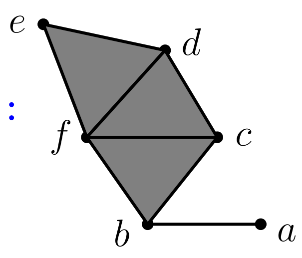

Poset homology and cohomology

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We will review some relevant concepts here by dealing directly with the chains of P. Let $\mathbf{k}$ be an arbitrary field or the ring of integers $\mathbb{Z}$. Define the *chain space*

$$C_j(P;\mathbf{k}) := \mathbf{k}\text{-module freely generated by } j\text{-chains of } P.$$

Then we can define *(reduced) chain* and *cochain complexes*

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The *homology* and the *cohomology* of P in dimension j is defined by

$$\tilde{H}_j(P;\mathbf{k}) := \ker\partial_j/\operatorname{im}\partial_{j+1}, \text{ and } \tilde{H}^j(P;\mathbf{k}) := \ker\delta_j/\operatorname{im}\delta_{j-1}.$$

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The chain and cochain spaces have been identified using the natural bases. This identification is given by the bilinear form $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ on $\bigoplus C_j(P;\mathbf{k})$ defined by $\langle\!\langle c, c' \rangle\!\rangle = \delta_{c,c'}$, where c, c' are chains of P, and extending by linearity.

Weighted Partition Poset

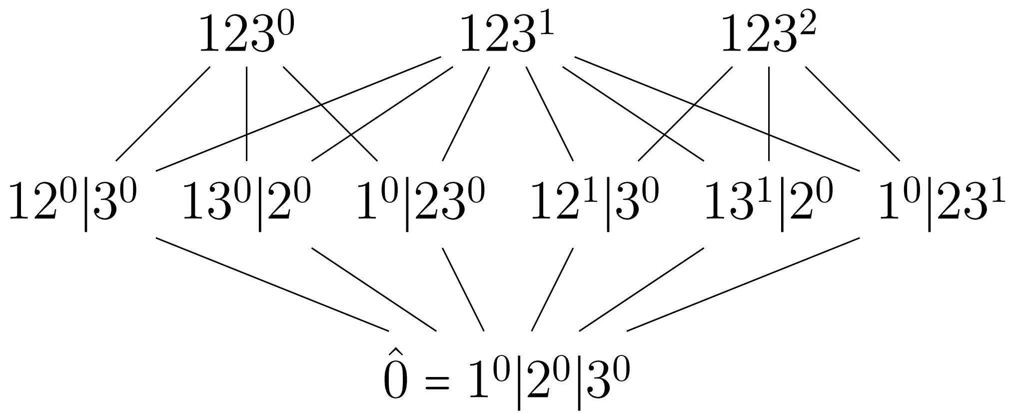
A *partition* of $[n] := \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets $\{B_1, \ldots, B_t\}$ of $[n]$ such that $\bigcup B_i = [n]$.

A *weighted partition* of $[n]$ is a set $\{B_1^{v_1}, \ldots, B_t^{v_t}\}$, where $\{B_1, \ldots, B_t\}$ is a partition of $[n]$, and $v_i \in \{0, 1, 2, \ldots, |B_i| - 1\}$ for all i. The *poset of weighted paritions of $[n]$*, denoted by Π_n^w, is the set of weighted paritions of $[n]$ with *covering relation* defined in the following way: An element $\sigma = \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\} \in \Pi_n^w$ is *covered* by another element $\pi \in \Pi_n^w$, i.e., $\sigma \lessdot \pi$, if there exist $1 \le i < j \le t$ and $\epsilon \in \{0, 1\}$ such that

$$\pi = \sigma \setminus \{B_i^{v_i}, B_j^{v_j}\} \cup \{(B_i \cup B_j)^{v_i + v_j + \epsilon}\}.$$

Weighted Partition Poset (cont'd)

Example. Π_3^w :



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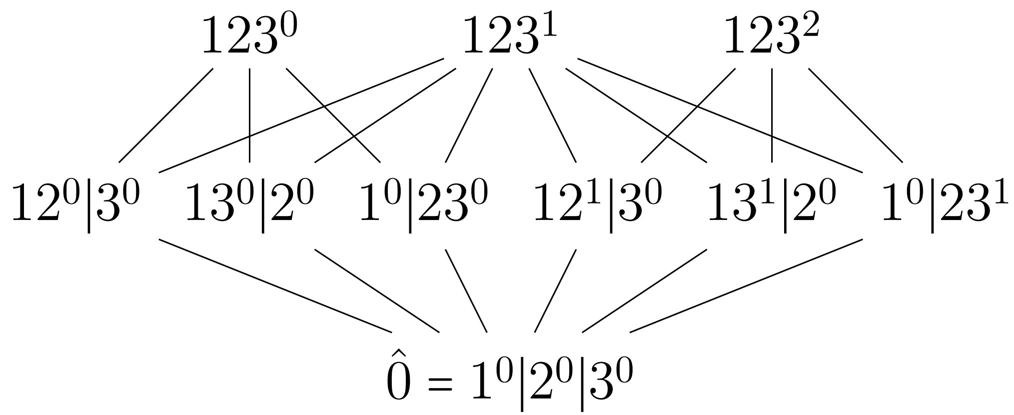

Note that Π_n^w has a unique minimal element $\hat{0} = 1^0|2^0|\cdots|n^0$, and n maximal elements $[n]^i$ for $0 \le i \le n-1$.

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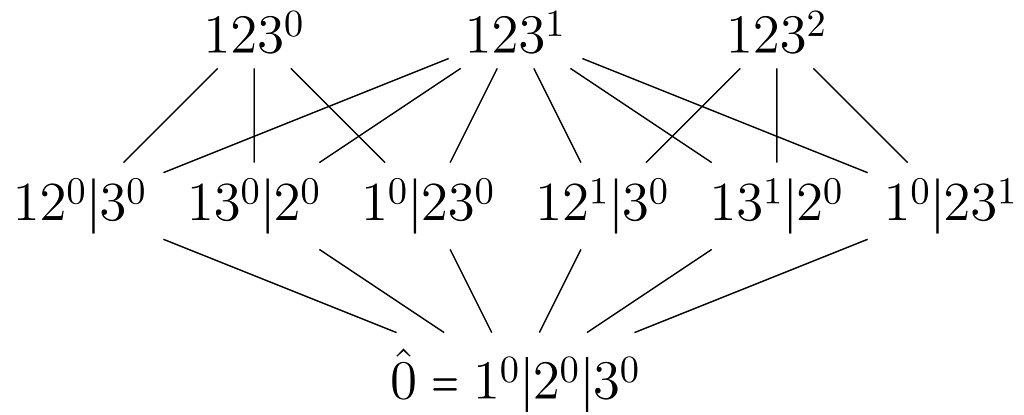

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The top (co)homology of open intervals $(\hat{0}, [n]^i)$ are studied.

$2v$-colored binary trees and $\mathscr{L}ie_2(n,i)$

A $2v$-*colored binary tree* on $[n]$ is a binary tree whose internal vertices are colored by red or blue, and leaves are labeled by $[n]$. We denote by $\mathcal{BT}_n$ the set of all $2v$-colored binary trees whose leaves are labeled by $[n]$, and by $\mathcal{BT}_{n,i}$ the set of trees in $\mathcal{BT}_n$ with i internal vertices being red.

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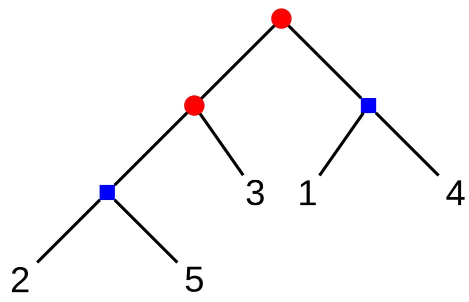
Example. A tree in $\mathcal{BT}_{5,2}$.



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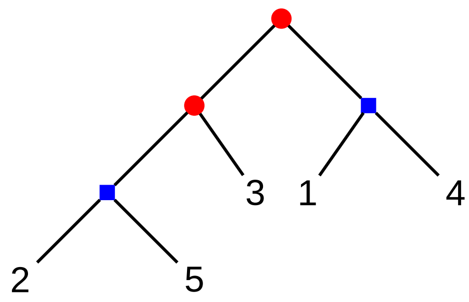


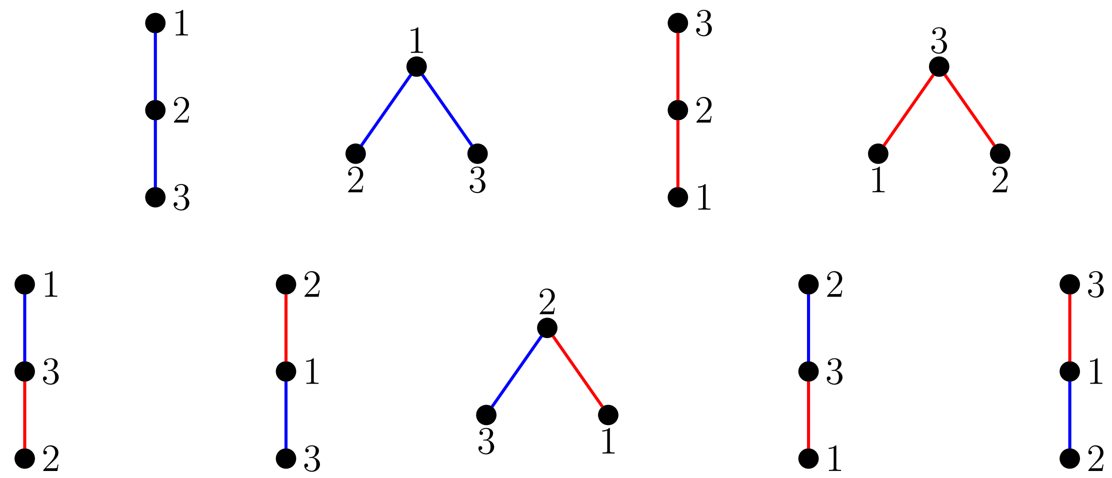
Let $\mathscr{L}ie_2(n,i)$ be the free $\mathbf{k}$-module generated by elements of $\mathcal{BT}_{n,i}$ subject to the antisymmetry and Jacobi relations.

Rooted Trees

Let $\mathcal{R}_n$ be the set of all the rooted trees on $[n]$. that is, all the rooted trees with n vertices that are labeled by $[n]$.

For any edge $\{i, j\}$ in a rooted tree, if i is the parent of j, we say $\{i, j\}$ an *increasing edge* if $i < j$, and a *decreasing edge* if $i > j$. For convenience, we color each increasing edge blue and each decreasing edge red. Let $\mathcal{R}_{n,i}$ be the set of rooted trees on $[n]$ that have i decreasing/red edges.

Example. When $n = 3$:



Relevant results on $(\hat{0}, [n]^i)$

- There are three bases known for $\mathscr{L}ie_2(n,i)$:
  - $\mathrm{Comb}^2_{n,i}$: Bershtein-Dotsenk-Khoroshkin introduced a comb basis, for $\mathscr{L}ie_2(n,i)$ generalizing Wach's comb basis for $\mathscr{L}ie(n)$.
  - $\mathrm{Liu}^2_{n,i}$: Liu introduced a Liu-Lyndon basis for $\mathscr{L}ie_2(n,i)$ generalizing the standard Lyndon basis.
  - $\mathrm{Lyn}^2_{n,i}$: González D'león-Wachs constructed another basis for $\mathscr{L}ie_2(n,i)$ that generalizes the standard Lyndon basis.

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- González D'león-Wachs showed that

$$\tilde{H}^{n-3}\left((\hat{0}, [n]^i)\right) \simeq_{\mathfrak{S}_n} \mathscr{L}ie_2(n,i) \otimes \mathrm{sgn}_n,$$

providing a way to construct bases for $\tilde{H}^{n-3}\left((\hat{0}, [n]^i)\right)$ from bases for $\mathscr{L}ie_2(n,i)$.

A question

There are three bases constructed for $\mathscr{L}ie_2(n,i)$ and $\tilde{H}^{n-3}\left((\hat{0},[n]^i)\right)$. We have

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González D'león-Wachs asked

Question 1. Are there nice bijections between $\mathcal{R}_{n,i}$ and $\mathrm{Comb}^2_{n,i}$ or $\mathrm{Lyn}^2_{n,i}$.

González D'león-Wachs's construction

- They associate $\bar{c}_T$, a maximal chain in $(\hat{0}, [n]^i)$, to each $T \in \mathcal{BT}_{n,i}$. Then for any basis $B \subseteq \mathcal{BT}_{n,i}$ of $\mathscr{L}ie_2(n,i)$,

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is a basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$.

Recall we have a bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ defined on the chain space of $(\hat{0}, [n]^i)$.

We say $T \in \mathcal{BT}_{n,i}$ and $G \in \mathcal{R}_{n,i}$ is a *good pair* if $\langle\!\langle \rho_G, \bar{c}_T \rangle\!\rangle \neq 0$.

An alternative definition

We say $T \in \mathcal{BT}_n$ and $G \in \mathcal{R}_n$ is a *good pair* if one of the following is satisfied:

(i) $n = 1$.

(ii) Suppose $n \geq 2$. Let T_1 and T_2 be the left and right subtrees of the root of T. There exists an edge e of G such that:

i. e has the same color as the root of T;

ii. By removing e from G, we obtain two rooted trees G_1 and G_2 satisfying T_i and G_i is a good pair for each i.

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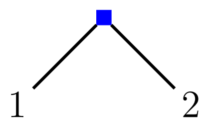

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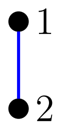

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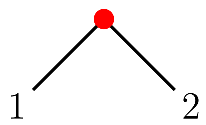

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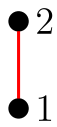

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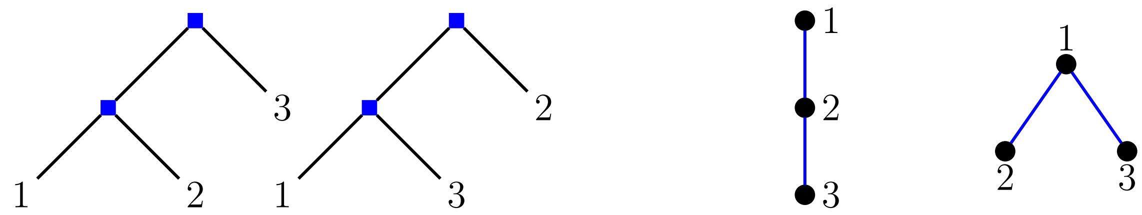
An alternative definition (cont'd)



T_1 T_2 G_1 G_2

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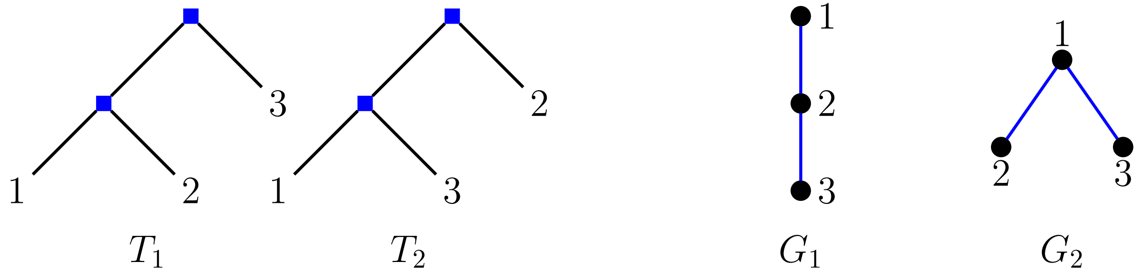

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Given $B \subseteq \mathcal{BT}_{n,i}$ a basis for $\mathscr{L}ie_2(n,i)$, we say a bijection $\psi : \mathcal{R}_{n,i} \to B$ is a *good-pair bijection* if G and $\psi(G)$ is a good pair for each $G \in \mathcal{R}_{n,i}$.

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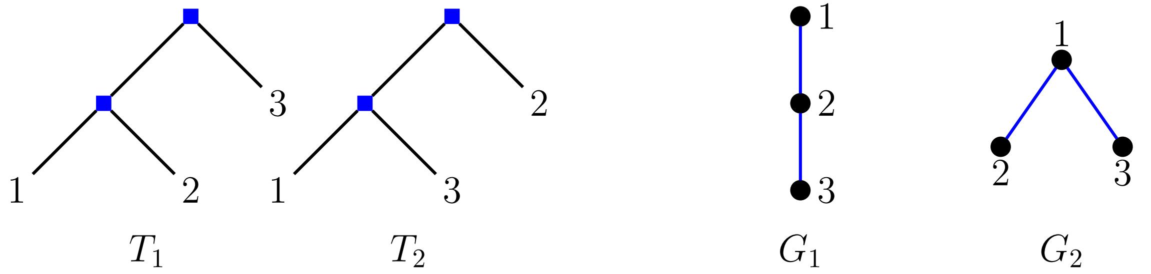

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Example. $B := \{T_1, T_2\}$ is a basis for $\mathscr{L}ie_2(3,0)$. It is clear there is a **unique** good-pair bijection from $\mathcal{R}_{3,0} = \{G_1, G_2\}$ to B :

$$G_1 \mapsto T_1, \quad G_2 \mapsto T_2.$$

A rephrased question and results

Recall there are three bases for $\mathscr{L}ie_2(n,i)$: $\mathrm{Comb}^2_{n,i}$, $\mathrm{Lyn}^2_{n,i}$ and $\mathrm{Liu}^2_{n,i}$.

Question 2. For each of the bases, do good-pair bijections exist? If so, is it unique?

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Conjecture 4. *There exists a unique good-pair bijection from* $\mathcal{R}_{n,i}$ *to* $\mathrm{Comb}^2_{n,i}$.

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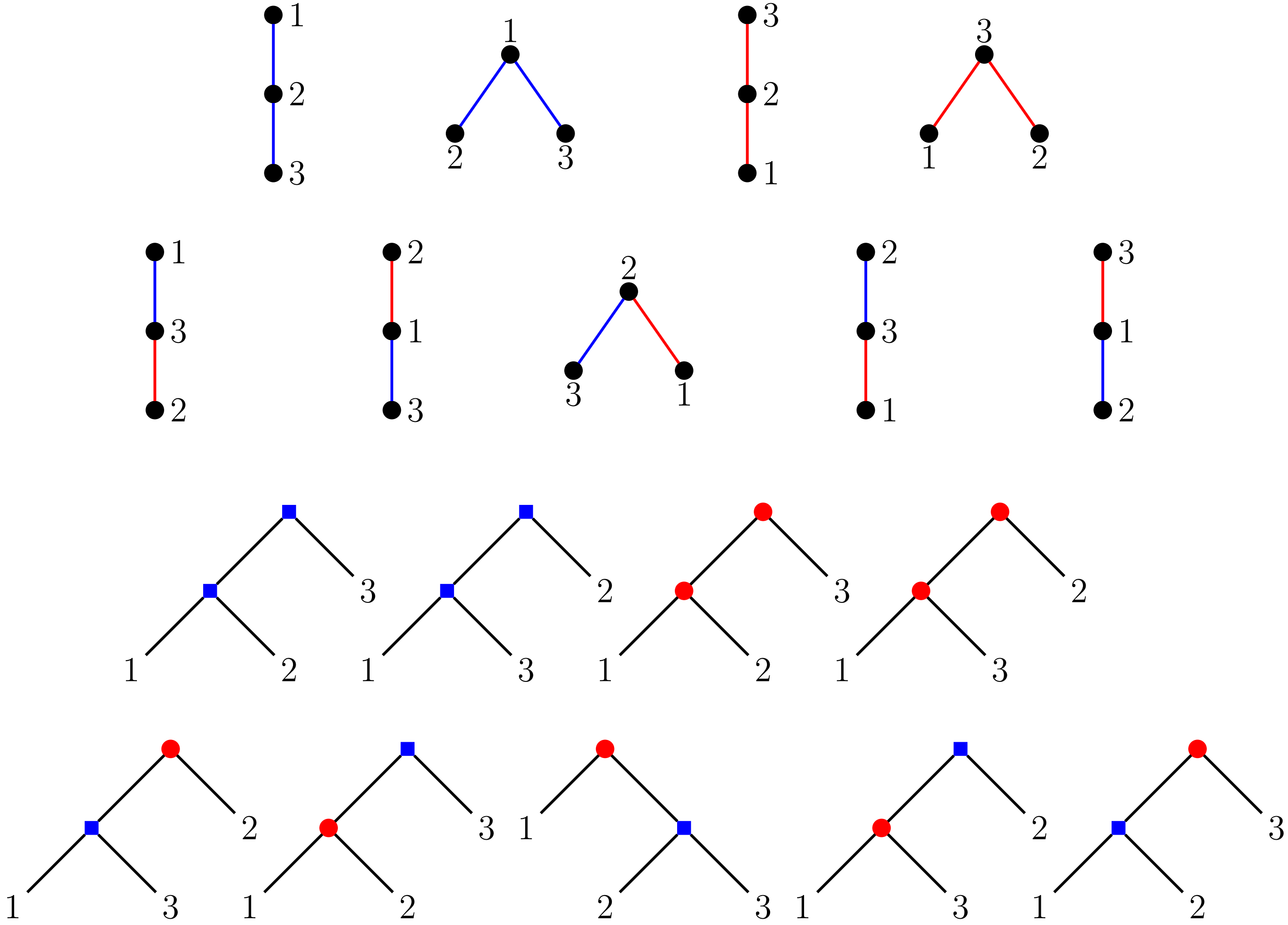
Results

Lemma 5. *Conjecture 4 are true for* $n \leq 4$.

Theorem 6. *Conjecture 4 are true when* $i = 0, n-1$.

The unique good-pair bijection from $\mathcal{R}_3$ to Comb_3^2



Idea of the proof

We prove the following stronger conjecture instead.

Conjecture 7. *There exist total orderings on* $\mathcal{R}_{n,i}$*:* $G_1, G_2, \dots, \dots$ *and* $\mathrm{Comb}^2_{n,i}$: $T_1, T_2, \dots$ *such that*

- G_i *and* T_i *is a good pair;*
- G_i *and* T_j *is not a good pair unless* $i \geq j$*.*