On bijections between rooted trees and the comb basis for the cohomology of the weighted partition poset

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Outline

- Introduction
  - Poset homology and cohomology
  - Known results on weighted partition poset
- Questions and results
  - González D’león-Wachs’ question
  - Discussion and rephrasing
  - Conjectures and results
Basic definitions

Given a poset (partially ordered set) $P$, a *chain* of $P$ is a totally ordered subset of $P$, and a *$j$-chain* is a chain of $j + 1$ elements.
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Given a poset (partially ordered set) $P$, a \textit{chain} of $P$ is a totally ordered subset of $P$, and a \textit{j-chain} is chain of $j + 1$ elements.

For any two comparable elements $x \leq y$ of $P$, we define

\[
[x, y] := \{z \in P \mid x \leq z \leq y\}; \\
(x, y) := \{z \in P \mid x < z < y\}.
\]
Given a poset (partially ordered set) $P$, a **chain** of $P$ is a totally ordered subset of $P$, and a **$j$-chain** is chain of $j + 1$ elements.

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$$[x, y] := \{z \in P \mid x \leq z \leq y\};$$

$$(x, y) := \{z \in P \mid x < z < y\}.$$

Given a poset $P$, the **order complex** of $P$, denoted by $\Delta(P)$ is an (abstract) simplicial complex, where the vertices are the elements of $P$ and the faces are the chains of $P$.

$$
\Delta(P) = \left\{ \emptyset, a, b, c, d, e, f, \right. \\
abla, ab, bc, bf, cd, cf, de, df, ef, \\
bcf, cdf, def \left. \right\}
$$

$||\Delta(P)||$:

- $a$ \\
- $b$ \\
- $c$ \\
- $d$ \\
- $e$ \\
- $\nabla$ \\
- $ab$ \\
- $ac$ \\
- $ad$ \\
- $ae$ \\
- $bc$ \\
- $bd$ \\
- $be$ \\
- $cd$ \\
- $ce$ \\
- $cf$ \\
- $de$ \\
- $df$ \\
- $e\nabla$ \\
- $abc$ \\
- $abd$ \\
- $abe$ \\
- $acd$ \\
- $ace$ \\
- $acf$ \\
- $ade$ \\
- $adf$ \\
- $aef$ \\
- $bcf$ \\
- $bc\nabla$ \\
- $bce$ \\
- $b\nabla$ \\
- $bcf$ \\
- $c\nabla$ \\
- $cdf$ \\
- $cde$ \\
- $cdf$ \\
- $def$ \\
- $e\nabla$ \\
- $\nabla\nabla$ \\
- $ac\nabla$ \\
- $ad\nabla$ \\
- $ae\nabla$ \\
- $bd\nabla$ \\
- $be\nabla$ \\
- $cd\nabla$ \\
- $ce\nabla$ \\
- $cf\nabla$ \\
- $de\nabla$ \\
- $df\nabla$ \\
- $e\nabla\nabla$ \\
- $\nabla\nabla\nabla$
Poset homology and cohomology

The (co)homology of $P$ is defined to be the reduced simplicial (co)homology of its order complex $\Delta(P)$. 
Poset homology and cohomology

The \textit{(co)homology} of $P$ is defined to be the reduced simplicial (co)homology of its order complex $\Delta(P)$.

We will review some relevant concepts here by dealing directly with the chains of $P$. Let $k$ be an arbitrary field or the ring of integers $\mathbb{Z}$. Define the \textit{chain space}

$$C_j(P; k) := k\text{-module freely generated by } j\text{-chains of } P.$$ 

Then we can define \textit{(reduced) chain} and \textit{cochain complexes}

$$\cdots \xleftarrow{\partial_{j-1}} C_{j+1}(P; k) \xrightarrow{\partial_{j+1}} C_j(P; k) \xleftarrow{\partial_j} C_{j-1}(P; k) \xrightarrow{\partial_{j-1}} \cdots$$

The \textit{homology} and the \textit{cohomology} of $P$ in dimension $j$ is defined by

$$\tilde{H}_j(P; k) := \ker \partial_j / \im \partial_{j+1}, \text{ and } \tilde{H}^j(P; k) := \ker \delta_j / \im \delta_{j-1}.$$
The **(co)homology** of $P$ is defined to be the reduced simplicial (co)homology of its order complex $\Delta(P)$.

We will review some relevant concepts here by dealing directly with the chains of $P$. Let $k$ be an arbitrary field or the ring of integers $\mathbb{Z}$. Define the *chain space*

$$C_j(P; k) := k\text{-module freely generated by } j\text{-chains of } P.$$  

Then we can define *(reduced) chain and cochain complexes*

$$\cdots \xrightarrow{\partial_{j+1}} C_{j+1}(P; k) \xleftarrow{\delta_{j+1}} C_j(P; k) \xrightarrow{\partial_j} C_{j-1}(P; k) \xleftarrow{\delta_j} \cdots$$

The **homology** and the **cohomology** of $P$ in dimension $j$ is defined by

$$\tilde{H}_j(P; k) := \ker \partial_j / \text{im } \partial_{j+1}, \text{ and } \tilde{H}^j(P; k) := \ker \delta_j / \text{im } \delta_{j-1}.$$ 

The chain and cochain spaces have been identified using the natural bases. This identification is given by the bilinear form $\langle \cdot, \cdot \rangle$ on $\bigoplus C_j(P; k)$ defined by $\langle c, c' \rangle = \delta_{c,c'}$, where $c, c'$ are chains of $P$, and extending by linearity.
A *partition* of $[n] := \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets $\{B_1, \ldots, B_t\}$ of $[n]$ such that $\bigcup B_i = [n]$.

A *weighted partition* of $[n]$ is a set $\{B_{v_1}^1, \ldots, B_{v_t}^t\}$, where $\{B_1, \ldots, B_t\}$ is a partition of $[n]$, and $v_i \in \{0, 1, 2, \ldots, |B_i| - 1\}$ for all $i$. The *poset of weighted partitions* of $[n]$, denoted by $\Pi^w_n$, is the set of weighted partitions of $[n]$ with *covering relation* defined in the following way: An element $\sigma = \{B_{v_1}^1, B_{v_2}^2, \ldots, B_{v_t}^t\} \in \Pi^w_n$ is *covered* by another element $\pi \in \Pi^w_n$, i.e., $\sigma \prec \pi$, if there exist $1 \leq i < j \leq t$ and $\epsilon \in \{0, 1\}$ such that

$$\pi = \sigma \setminus \{B_{v_i}^i, B_{v_j}^j\} \cup \{(B_i \cup B_j)^{v_i+v_j+\epsilon}\}.$$
Example. $\Pi^w_3$:

\[
\begin{array}{c}
\Pi^w_3 :\\
\begin{array}{c}
123^0 \\
12^0|3^0 \\
\hat{0} = 1^0|2^0|3^0 \\
13^0|2^0 \\
1^0|2^3 \\
12^1|3^0 \\
13^1|2^0 \\
1^0|23^1 \\
123^1 \\
12^1|3^1 \\
13^1|2^1 \\
1^0|23^1 \\
123^2 \\
12^2|3^2 \\
13^2|2^2 \\
1^0|23^2 \\
\end{array}
\end{array}
\]
Example. \( \Pi^w_3 \): 

\[
\begin{array}{c}
123^0 \\
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\end{array}
\end{array}
\]

Note that \( \Pi^w_n \) has a unique minimal element \( \hat{0} = 1^0|2^0|\cdots|n^0 \), and \( n \) maximal elements \([n]^i\) for \( 0 \leq i \leq n - 1 \).
Example. $\Pi_3^w$:

```
\hat{0} = 1^0|2^0|3^0
```

Note that $\Pi_n^w$ has a unique minimal element $\hat{0} = 1^0|2^0|\cdots|n^0$, and $n$ maximal elements $[n]^i$ for $0 \leq i \leq n - 1$.

The top (co)homology of open intervals $(\hat{0}, [n]^i)$ are studied.
A $2v$-colored binary tree on $[n]$ is a binary tree whose internal vertices are colored by red or blue, and leaves are labeled by $[n]$. We denote by $\mathcal{BT}_n$ the set of all $2v$-colored binary trees whose leaves are labeled by $[n]$, and by $\mathcal{BT}_{n,i}$ the set of trees in $\mathcal{BT}_n$ with $i$ internal vertices being red.
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\textbf{Example.} A tree in $\mathcal{BT}_{5,2}$.
A 2v-colored binary tree on \([n]\) is a binary tree whose internal vertices are colored by red or blue, and leaves are labeled by \([n]\). We denote by \(BT_n\) the set of all 2v-colored binary trees whose leaves are labeled by \([n]\), and by \(BT_{n,i}\) the set of trees in \(BT_n\) with \(i\) internal vertices being red.

**Example.** A tree in \(BT_{5,2}\).

![Binary tree diagram]

Let \(\text{Lie}_2(n,i)\) be the free \(k\)-module generated by elements of \(BT_{n,i}\) subject to the antisymmetry and Jacobi relations.
Let $\mathcal{R}_n$ be the set of all the rooted trees on $[n]$. that is, all the rooted trees with $n$ vertices that are labeled by $[n]$.

For any edge $\{i, j\}$ in a rooted tree, if $i$ is the parent of $j$, we say $\{i, j\}$ an *increasing edge* if $i < j$, and a *decreasing edge* if $i > j$. For convenience, we color each increasing edge blue and each decreasing edge red. Let $\mathcal{R}_{n,i}$ be the set of rooted trees on $[n]$ that have $i$ decreasing/red edges.

**Example.** When $n = 3$:

1. 1
   - 2
   - 3
2. 1
   - 2
   - 3
3. 3
   - 2
   - 1
4. 3
   - 1
   - 2
5. 3
   - 3
   - 1
   - 2
6. 3
   - 1
   - 2
   - 3
There are three bases known for \( \mathcal{L}ie_2(n, i) \):

- \( \text{Comb}^2_{n,i} \): Bershtein-Dotsenk-Khoroshkin introduced a comb basis, for \( \mathcal{L}ie_2(n, i) \) generalizing Wach's comb basis for \( \mathcal{L}ie(n) \).
- \( \text{Liu}^2_{n,i} \): Liu introduced a Liu-Lyndon basis for \( \mathcal{L}ie_2(n, i) \) generalizing the standard Lyndon basis.
- \( \text{Lyn}^2_{n,i} \): González D’león-Wachs constructed another basis for \( \mathcal{L}ie_2(n, i) \) that generalizes the standard Lyndon basis.
On bijections between rooted trees and the comb basis

Relevant results on \((\mathcal{C}, [n])\)

- There are three bases known for \(\mathcal{L}ie_2(n, i)\):
  - \(\text{Comb}^2_{n, i}\): Bershtein-Dotsenk-Khoroshkin introduced a comb basis, for \(\mathcal{L}ie_2(n, i)\) generalizing Wach’s comb basis for \(\mathcal{L}ie(n)\).
  - \(\text{Liu}^2_{n, i}\): Liu introduced a Liu-Lyndon basis for \(\mathcal{L}ie_2(n, i)\) generalizing the standard Lyndon basis.
  - \(\text{Lyn}^2_{n, i}\): González D’león-Wachs constructed another basis for \(\mathcal{L}ie_2(n, i)\) that generalizes the standard Lyndon basis.

- Liu and Dotsenko-Khoroshkin showed that

\[
\text{rank } \mathcal{L}ie_2(n, i) = |\mathcal{R}_{n, i}|.
\]
Relevant results on $(\hat{0}, [n]^i)$

- There are three bases known for $\mathcal{L}ie_2(n, i)$:
  - $\text{Comb}^2_{n,i}$: Bershtein-Dotsenk-Khoroshkin introduced a comb basis, generalizing Wach's comb basis for $\mathcal{L}ie(n)$.
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  - $\text{Lyn}^2_{n,i}$: González D'león-Wachs constructed another basis that generalizes the standard Lyndon basis.

- Liu and Dotsenko-Khoroshkin showed that
  $$\text{rank } \mathcal{L}ie_2(n, i) = |\mathcal{R}_{n,i}|.$$

- González D'león-Wachs showed that
  $$\tilde{H}^{n-3}((\hat{0}, [n]^i)) \cong_{\mathfrak{S}_n} \mathcal{L}ie_2(n, i) \otimes \text{sgn}_n,$$
  providing a way to construct bases for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ from bases for $\mathcal{L}ie_2(n, i)$. 
A question

There are three bases constructed for $\mathcal{L}ie_2(n, i)$ and $\tilde{H}^{n-3}((\hat{0}, [n]^i))$. We have

$$|\text{Comb}_{n,i}^2| = |\text{Liu}_{n,i}^2| = |\text{Lyn}_{n,i}^2| = |\mathcal{R}_{n,i}|.$$
A question

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$$|\text{Comb}_{n,i}^2| = |\text{Liu}_{n,i}^2| = |\text{Lyn}_{n,i}^2| = |\mathcal{R}_{n,i}|.$$  

Liu gave a bijection between $\text{Liu}_{n,i}^2$ and $\mathcal{R}_{n,i}$. 
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González D'león constructed a bijection between $\text{Comb}^2_{n,i}$ and $\text{Lyn}^2_{n,i}$. 
A question

There are three bases constructed for $\mathcal{L}ie_2(n,i)$ and $\tilde{H}^{n-3}\left((\hat{0}, [n]^i)\right)$. We have

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González D’león constructed a bijection between $\text{Comb}_{n,i}^2$ and $\text{Lyn}_{n,i}^2$.

González D’león-Wachs asked

**Question 1.** Are there nice bijections between $\mathcal{R}_{n,i}$ and $\text{Comb}_{n,i}^2$ or $\text{Lyn}_{n,i}^2$.
They associate $\bar{c}_T$, a maximal chain in $(\hat{0}, [n]^i)$, to each $T \in \mathcal{BT}_{n,i}$. Then for any basis $B \subseteq \mathcal{BT}_{n,i}$ of $\mathcal{L}ie_2(n,i)$,

$$\{\bar{c}_T \mid T \in B\}$$

is a basis for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$. 
On bijections between rooted trees and the comb basis

**González D’león-Wachs’s construction**

- They associate \( \tilde{c}_T \), a maximal chain in \((\hat{0}, [n]^i)\), to each \( T \in B\mathcal{T}_{n,i} \). Then for any basis \( B \subseteq B\mathcal{T}_{n,i} \) of \( \mathcal{L}i_2(n,i) \),

  \[
  \{\tilde{c}_T \mid T \in B\}
  \]

  is a basis for \( \tilde{H}^{n-3}((\hat{0}, [n]^i)) \).

- They associate \( \rho_G \), a fundamental cycle of the spherical complex \( \Delta(\overline{\Pi}_G) \), to each \( G \in \mathcal{R}_{n,i} \), and show that

  \[
  \{\rho_G \mid G \in \mathcal{R}_{n,i}\}
  \]

  is a basis for \( \tilde{H}_{n-3}((\hat{0}, [n]^i)) \).
They associate $\bar{c}_T$, a maximal chain in $(\hat{0}, [n]^i)$, to each $T \in BT_{n,i}$. Then for any basis $B \subseteq BT_{n,i}$ of $Lie_2(n,i)$,

$$\{\bar{c}_T | T \in B\}$$

is a basis for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$.

They associate $\rho_G$, a fundamental cycle of the spherical complex $\Delta(\bar{\Pi}_G)$, to each $G \in R_{n,i}$, and show that

$$\{\rho_G | G \in R_{n,i}\}$$

is a basis for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$.

Recall we have a bilinear form $\langle \cdot , \cdot \rangle$ defined on the chain space of $(\hat{0}, [n]^i)$.

We say $T \in BT_{n,i}$ and $G \in R_{n,i}$ is a **good pair** if $\langle \rho_G, \bar{c}_T \rangle \neq 0$. 
An alternative definition

We say $T \in B\mathcal{T}_n$ and $G \in R_n$ is a **good pair** if one of the following is satisfied:

(i) $n = 1$.

(ii) Suppose $n \geq 2$. Let $T_1$ and $T_2$ be the left and right subtrees of the root of $T$. There exists an edge $e$ of $G$ such that:
   i. $e$ has the same color as the root of $T$;
   ii. By removing $e$ from $G$, we obtain two rooted trees $G_1$ and $G_2$ satisfying $T_i$ and $G_i$ is a good pair for each $i$. 
An alternative definition

We say $T \in \mathcal{B}T_n$ and $G \in \mathcal{R}_n$ is a **good pair** if one of the following is satisfied:

(i) $n = 1$.

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- i. $e$ has the same color as the root of $T$;
- ii. By removing $e$ from $G$, we obtain two rooted trees $G_1$ and $G_2$ satisfying $T_i$ and $G_i$ is a good pair for each $i$.

**Example.** When $n = 2$:

- $1 \quad 2$   and   $1 \quad 2$ is a good pair
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Example.

- $G_1$ is a good pair with $T_1$, but not with $T_2$.
- $G_2$ is a good pair with both $T_1$ and $T_2$. 
An alternative definition (cont’d)

Example.

- $G_1$ is a good pair with $T_1$, but not with $T_2$.
- $G_2$ is a good pair with both $T_1$ and $T_2$.

Given $B \subseteq B_{T_{n,i}}$ a basis for $\text{Lie}_2(n, i)$, we say a bijection $\psi : R_{n,i} \to B$ is a good-pair bijection if $G$ and $\psi(G)$ is a good pair for each $G \in R_{n,i}$. 
An alternative definition (cont’d)

Example.

- \(G_1\) is a good pair with \(T_1\), but not with \(T_2\).
- \(G_2\) is a good pair with both \(T_1\) and \(T_2\).

Given \(B \subseteq B \mathcal{T}_{n,i}\) a basis for \(\mathcal{L}ie_2(n,i)\), we say a bijection \(\psi : \mathcal{R}_{n,i} \rightarrow B\) is a **good-pair bijection** if \(G\) and \(\psi(G)\) is a good pair for each \(G \in \mathcal{R}_{n,i}\).

Example. \(B := \{T_1, T_2\}\) is a basis for \(\mathcal{L}ie_2(3,0)\). It is clear there is a **unique** good-pair bijection from \(\mathcal{R}_{3,0} = \{G_1, G_2\}\) to \(B\) :

\[G_1 \mapsto T_1, \quad G_2 \mapsto T_2.\]
Recall there are three bases for $\mathcal{L}ie_2(n, i)$: $\text{Comb}_n^2$, $\text{Lyn}_n^2$, and $\text{Liu}_n^2$.

**Question 2.** For each of the bases, do good-pair bijections exist? If so, is it unique?
Recall there are three bases for $\mathcal{L}ie_2(n, i)$: $\text{Comb}^2_{n,i}$, $\text{Lyn}^2_{n,i}$ and $\text{Liu}^2_{n,i}$.

**Question 2.** For each of the bases, do good-pair bijections exist? If so, is it unique?

**Fact 3.** The bijection given from $\mathcal{R}_{n,i}$ to $\text{Liu}^2_{n,i}$ is a good-pair bijection, and it is unique.
Recall there are three bases for $\mathcal{L}ie_2(n, i) : \text{Comb}^2_{n,i}$, $\text{Lyn}^2_{n,i}$ and $\text{Liu}^2_{n,i}$.

**Question 2.** For each of the bases, do good-pair bijections exist? If so, is it unique?

**Fact 3.** The bijection given from $\mathcal{R}_{n,i}$ to $\text{Liu}^2_{n,i}$ is a good-pair bijection, and it is unique.

**Conjecture 4.** There exists a unique good-pair bijection from $\mathcal{R}_{n,i}$ to $\text{Comb}^2_{n,i}$.
Recall there are three bases for $\mathcal{L}ie_2(n, i)$: $\text{Comb}_{n,i}^2$, $\text{Lyn}_{n,i}^2$, and $\text{Liu}_{n,i}^2$.

**Question 2.** For each of the bases, do good-pair bijections exist? If so, is it unique?

**Fact 3.** The bijection given from $\mathcal{R}_{n,i}$ to $\text{Liu}_{n,i}^2$ is a good-pair bijection, and it is unique.

**Conjecture 4.** There exists a unique good-pair bijection from $\mathcal{R}_{n,i}$ to $\text{Comb}_{n,i}^2$.

**Results**

**Lemma 5.** Conjecture 4 are true for $n \leq 4$.

**Theorem 6.** Conjecture 4 are true when $i = 0, n - 1$. 
The unique good-pair bijection from $\mathcal{R}_3$ to $\text{Comb}_3^2$
Idea of the proof

We prove the following stronger conjecture instead.

**Conjecture 7.** There exist total orderings on $\mathcal{R}_{n,i} : G_1, G_2, \ldots, \ldots$ and $\text{Comb}^2_{n,i} : T_1, T_2, \ldots$ such that

- $G_i$ and $T_i$ is a good pair;
- $G_i$ and $T_j$ is not a good pair unless $i \geq j$. 