The volume of the Birkhoff polytope

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The volume of the Birkhoff polytope

Outline

- Introduction to the Birkhoff polytope
- Our formula for the volume of the Birkhoff polytope
- Theories behind the proof
The volume of the Birkhoff polytope

Birkhoff polytope and its faces

**Definition 1.** The *Birkhoff polytope*, $B_n$, is the set of all *doubly-stochastic matrices*, that is, the real nonnegative matrices with all row and column sums equal to one.

We consider $B_n$ in the $n^2$-dimensional space $\mathbb{R}^{n^2} = \{ n \times n \text{ real matrices} \}$.

Faces of $B_n$:

- The vertices of $B_n$ are the $n \times n$ permutation matrices. The vertex set $V(B_n)$ can be considered as $S_n$, the set of all the permutations on $[n] = \{1, 2, \ldots, n\}$.

- $B_n$ has $n^2$ facets: for each pair of $(i, j)$ with $1 \leq i, j \leq n$, the doubly-stochastic matrices with $(i, j)$th entry equal to 0 is a facet.

- In general, if $S$ is the union of the nonzero indices of a set of permutation matrices, then the set of doubly-stochastic matrices $M$ with $M(i, j) = 0$ for $(i, j) \notin S$ is a face of $B_n$, and every face arises this way.
Example: faces of $B_3$

$S = \begin{pmatrix} \star & 0 & 0 \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence $B_3 \cap \{M \mid M(1, 2) = M(1, 3) = M(2, 1) = M(2, 3) = 0\}$ is a face of $B_3$. 
The volume of the Birkhoff polytope

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- $S = \begin{pmatrix} 0 & \star & 0 \\ 0 & 0 & \star \\ \star & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Hence $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (123)$ is a vertex of $B_3$. 
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Example: faces of $B_3$

\[
S = \begin{pmatrix}
\ast & 0 & \ast \\
\ast & \ast & \ast \\
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\end{pmatrix}.
\]

$B_3 \cap \{M \mid M(1, 2) = 0\}$ is a facet of $B_3$. 
Why do we care about the volume of $B_n$? It turns out if we can calculate the volumes of each face of $B_n$, then one can generate doubly-stochastic matrices uniformly at random.
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Some work on the volume of $B_n$ and its faces:

- Bona gave a lower bound for $\text{Vol}(B_n)$.
- A special face of $B_n$: the volume of the Chan-Robbins-Yen polytope is a product of Catalan numbers.
- Beck-Pixton calculated $\text{Vol}(B_n)$ up to $n = 10$. 

The volume of $B_n$
Our result

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Our goal is to give a combinatorial formula for \( \text{Vol}(B_n) \).

We present closed formulas for all coefficients of the Ehrhart polynomial \( i(B_n, m) \).

In particular, the volume of \( B_n \) looks like this

\[
\text{Vol}(B_n) = \frac{1}{(n-1)^2!} \sum_{\sigma \in S_n=V(B_n)} \sum_{\gamma_1(c, \sigma) \cdots \gamma_{(n-1)^2}(c, \sigma)} \langle c, \sigma \rangle^{(n-1)^2}.
\]

The rational function summands are indexed by combinatorial data.
Arborescences

- A directed spanning tree with all edges pointing away from the root is an arborescence.
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• The set of all arborescences on the nodes $[n] = \{1, 2, \ldots, n\}$ will be denoted by $\text{Arb}(n)$. It is well known that the cardinality of $\text{Arb}(n)$ is $n^{n-2}$.
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• For any $T \in \text{Arb}(n)$, we denote by $E(T)$ the set of edges of $T$. The cardinality of $E(T)$ is $n - 1$. Hence, there are $(n - 1)^2$ (directed) edges not in $E(T)$.
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- For any $e \notin E(T)$, there is a unique oriented cycle, denoted by $\text{cycle}(T + e)$, in $T + e$. We associate a $n \times n (0, -1, 1)$-matrix $W_{T,e}$ to it.
\( W^{T,e} \): The matrix associated to \( cycle(T + e) \)

\begin{align*}
\hat{W}^{T_A,(1,3)} &= \begin{pmatrix}
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 1 & -1
\end{pmatrix}.
\end{align*}

- The edges in \( cycle(T, e) \) all have the same orientation: choose \( T_A \) and \( e = (1,3) \),

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- There are two vertices \( s, t \) in \( cycle(T, e) \) such that all the edges are directed from \( s \) to \( t \): choose \( T_C \) and \( e = (2, 1) \),

\[ W^{T_C,(2,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}. \]
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**Our main result**

**Theorem 2** (DeLoera–L–Yoshida). The volume of the Birkhoff polytope $B_n$ is given by the formula

$$
\text{Vol}(B_n) = \frac{1}{((n-1)^2)!} \sum_{\sigma \in S_n} \sum_{T \in \text{Arb}(n)} \frac{\langle c, \sigma \rangle^{(n-1)^2}}{\prod_{e \notin E(T)} \langle c, W^{T,e} \sigma \rangle}.
$$

In the formula $c \in \mathbb{R}^{n^2}$ is any vector such that $\langle c, W^{T,e} \sigma \rangle$ is non-zero for all pairs $(T, e)$ of an arborescence $T$ and an edge $e \notin E(T)$ and all $\sigma \in S_n$.

We can for instance choose $c = (1, 2, 4, 8, \ldots, 2^{n^2-1})$ to obtain a completely explicit formula.
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**Definition 3.** For any polytope $P \subset \mathbb{R}^d$ and some positive integer $m \in \mathbb{N}$, the \textit{$m$th dilated polytope} of $P$ is $mP = \{mx : x \in P\}$. We denote by

$$i(m, P) = |mP \cap \mathbb{Z}^d|$$

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**Example:**

![Diagram of lattice points and polytopes](image)
**Volume/area and lattice points enumerator**

**Pick’s theorem:** For any integral polygon $Q$:

$$\text{area}(Q) = |Q \cap \mathbb{Z}^2| - \frac{1}{2}|\partial(Q) \cap \mathbb{Z}^2| - 1.$$
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If $P$ is an integral polygon, then so is $mP$.

$$i(P, m) = \text{area}(mP) + \frac{1}{2} |\partial(mP) \cap \mathbb{Z}^d| + 1$$

$$= \text{area}(P)m^2 + \frac{1}{2} |\partial(P) \cap \mathbb{Z}^d|m + 1$$
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**Ehrhart’s theorem:** Let $P$ be a $d$-dimensional integral polytope, then $i(P, m)$ is a polynomial in $m$ of degree $d$, with the leading coefficient the volume $\text{Vol}(P)$ of $P$.

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of $P$. 
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**Multivariate generating function**

For any polyhedron $P \in \mathbb{R}^d$, we define the *multivariate generating function* (MGF) of $P$ as

$$f(P, z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^\alpha,$$

where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$.

One sees that by setting $z = (1, 1, \ldots, 1)$, we get the number of lattice points in $P$ if $P$ is a polytope.

**Example:** Let $P$ be the polytope with vertices $v_1 = (0, 0), v_2 = (2, 0)$ and $v_3 = (0, 2)$.

$$P: \begin{array}{ccc}
& (0,2) & \\
(0,1) & & (1,1) \\
& (0,0) & (1,0) & (2,0)
\end{array}$$

$$f(P, z) = z_1^0 z_2^0 + z_1^1 z_2^0 + z_1^0 z_2^1 + z_1^1 z_2^1 + z_1^0 z_2^2 = 1 + z_1 + z_2 + z_1 z_2 + z_2^2.$$
It turns out that \( f(P, z) \) can be written as a rational function, for any rational polyhedron \( P \).

**Lemma 4** (Brion, 1988; Lawrence, 1991). *Let \( P \) be a rational polyhedron and let \( V(P) \) be the vertex set of \( P \). Then, considered as rational functions,

\[
f(P, z) = \sum_{v \in V(P)} f(C(P, v), z),
\]

where \( C(P, v) \) is the **supporting cone** of \( P \) at \( v \), i.e., the smallest cone with vertex \( v \) containing \( P \).
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**Remark:** We only need to find the MGF for one of the vertices of $B_n$, then apply the action of symmetric group to get the others. We will do this at the vertex associated to the identity permutation matrix, denoted by $I$. We denote by $C_n$ the supporting cone of $B_n$ at $I$.

Our goal is then to find $f(C_n, z)$. 
If \( K \) is a \( d \)-dimensional cone in \( \mathbb{R}^e \), generated by vectors \( \{r_i\}_{1 \leq i \leq d} \) such that the \( r_i \)'s form a \( \mathbb{Z} \)-basis of the lattice \( \text{span}(\{r_i\}) \cap \mathbb{Z}^e \), then we say \( K \) is a \textit{unimodular cone}.

\textbf{Lemma 5.} \textit{If} \( K \) \textit{is a} \( d \)-\textit{dimensional unimodular cone at an integral vertex} \( v \) \textit{generated by the vectors} \( \{r_i\}_{1 \leq i \leq d} \), \textit{then we have}

\[
f(K, z) = z^v \prod_{i=1}^{d} \frac{1}{1 - z^{r_i}}.
\]
Example of the lemmas
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Recall that $f(P, z) = 1 + z_1 + z_1^2 + z_2 + z_1z_2 + z_2^2$. 

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A unimodular cone generated by vectors $r_1 = (1, 0)$ and $r_2 = (0, 1)$.

\[
f(C(P, v_1), z) = z^{(0,0)} \prod_{i=1}^{2} \frac{1}{1-z^r_i} = \frac{1}{(1-z_1)(1-z_2)}.
\]
Example: Let $P$ be the polytope with vertices $v_1 = (0, 0)$, $v_2 = (2, 0)$ and $v_3 = (0, 2)$.

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A unimodular cone generated by vectors $r_1 = (-1, 0)$ and $r_2 = (-1, 1)$.

$$f(C(P, v_2), z) = z^{(2,0)} \prod_{i=1}^{2} \frac{1}{1 - z^{r_i}} = \frac{z_1^2}{(1-z_1^{-1})(1-z_1^{-1}z_2)} = \frac{z_1^4}{(z_1-1)(z_1-z_2)}.$$
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Recall that $f(P, z) = 1 + z_1 + z_1^2 + z_2 + z_1z_2 + z_2^2$.

\begin{align*}
C(P, v_1) & : \\
& \text{A unimodular cone generated by vectors } r_1 = (1, 0) \text{ and } r_2 = (0, 1). \\
& f(C(P, v_1), z) = z^{(0,0)} \prod_{i=1}^{2} \frac{1}{1-z_i} = \frac{1}{(1-z_1)(1-z_2)}.
\end{align*}

\begin{align*}
C(P, v_2) & : \\
& \text{A unimodular cone generated by vectors } r_1 = (-1, 0) \text{ and } r_2 = (-1, 1). \\
& f(C(P, v_2), z) = z^{(2,0)} \prod_{i=1}^{2} \frac{1}{1-z_i} = \frac{z_1^2}{(1-z_1^{-1})(1-z_1z_2^{-1})} = \frac{z_1^4}{(z_1-1)(z_1-z_2)}.
\end{align*}

\begin{align*}
C(P, v_3) & : \\
& \text{A unimodular cone generated by vectors } r_1 = (0, -1) \text{ and } r_2 = (1, -1). \\
& f(C(P, v_3), z) = z^{(0,2)} \prod_{i=1}^{2} \frac{1}{1-z_i} = \frac{z_2^2}{(1-z_2^{-1})(1-z_1z_2^{-1})} = \frac{z_2^4}{(z_2-1)(z_2-z_1)}.
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Example: Let $P$ be the polytope with vertices $v_1 = (0, 0), v_2 = (2, 0)$ and $v_3 = (0, 2)$.

Recall that $f(P, z) = 1 + z_1 + z_1^2 + z_2 + z_1 z_2 + z_2^2$.

$$
\sum_{i=1}^{3} f(C(P, v_i), z) = \frac{(z_1-z_2)-z_1^4(1-z_2)+z_2^4(1-z_1)}{(1-z_1)(1-z_2)(z_1-z_2)} = 1 + z_1 + z_1^2 + z_2 + z_1 z_2 + z_2^2 = f(P, z).
$$
Barvinok’s algorithm

- Barvinok gave an algorithm to decompose a cone $C'$ as a signed sum of simple unimodular cones.

- Using the Brion’s polarization trick, we can ignore the lower dimensional cones. This trick involves using the dual cone of $C'$ instead.
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- Barvinok gave an algorithm to decompose a cone $C$ as a signed sum of simple unimodular cones.
- Using the Brion’s polarization trick, we can ignore the lower dimensional cones. This trick involves using the *dual cone* of $C$ instead.

**Algorithm:** Input a cone $C$ with vertex $v$

1. Find a dual cone $K$ to $C$.
2. Apply the Barvinok decomposition to $K$ and get a set of signed unimodular cones $K_i$.
3. Find dual cone $C_i$ of each $K_i$. (Note $C_i$ is unimodular as well.)
4. $f(C, z) = \sum_i \epsilon_i f(C_i, z)$, where $\epsilon_i$ is the sign of $C_i$. 
Apply the algorithm to $C_n$

For step (ii) in the algorithm, we show that any triangulation of the dual cone of $C_n$ gives a set of unimodular cones. Therefore, instead of using Barvinok’s method, we use the idea of Gröbner bases of toric ideals to produce triangulations.
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The multivariate generating function of $C_n$ is given by

$$f(C_n, z) = \sum_{T \in \text{Arb}(\ell, n)} z^I \prod_{e \notin E(T)} \frac{1}{(1 - \prod_z z^{w^{T,e}})}.$$
The MGF of the dilation $mB_n$

We get the multivariate generating function of $B_n$:

$$f(B_n, z) = \sum_{\sigma \in S_n} \sum_{T \in \text{Arb}(\ell, n)} z^{\sigma} \prod_{e \notin E(T)} \frac{1}{(1 - \prod z^{WT,e}^{T,\sigma})}.$$
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**Theorem 6** (DeLoera–L–Yoshida). The multivariate generating function of $mB_n$ is given by

$$f(mB_n, z) = \sum_{\sigma \in S_n} \sum_{T \in \text{Arb}(\ell,n)} z^{m\sigma} \prod_{e \notin E(T)} \frac{1}{(1 - \prod z^{W_T,e \sigma})},$$
Corollary 7. The Ehrhart polynomial $i(B_n, m)$ of $B_n$ is given by the formula

$$i(B_n, m) = \sum_{k=0}^{(n-1)^2} m^k \frac{1}{k!} \sum_{\sigma \in S_n} \sum_{T \in \text{Arb}(\ell, n)} \frac{(\langle c, \sigma \rangle)^k \text{td}_{(n-1)^2-k}(\{\langle c, W^T, e \sigma \rangle, e \notin E(T)\})}{\prod_{e \notin E(T)} \langle c, W^T, e \sigma \rangle}.$$ 

The symbol $\text{td}_j(S')$ is the $j$-th Todd polynomial evaluated at the numbers in the set $S'$. The vector $c \in \mathbb{R}^{n^2}$ is any vector such that $\langle c, W^T, e \sigma \rangle$ is non-zero for all pairs $(T, e)$ of an $\ell$-arborescence $T$ and an arc $e \notin E(T)$ and all $\sigma \in S_n$. 
Corollary 7. The Ehrhart polynomial \(i(B_n, m)\) of \(B_n\) is given by the formula

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The symbol \(\text{td}_j(S)\) is the \(j\)-th Todd polynomial evaluated at the numbers in the set \(S\). The vector \(c \in \mathbb{R}^{n^2}\) is any vector such that \(\langle c, W^T_e \sigma \rangle\) is non-zero for all pairs \((T, e)\) of an \(\ell\)-arborescence \(T\) and an arc \(e \notin E(T)\) and all \(\sigma \in S_n\).

As a special case, the normalized volume of \(B_n\) is given by

\[
\text{Vol}(B_n) = \frac{1}{((n-1)^2)!} \sum_{\sigma \in S_n} \sum_{T \in \text{Arb}(\ell, n)} \frac{(\langle c, \sigma \rangle)^{(n-1)^2}}{\prod_{e \notin E(T)} \langle c, W^T_e \sigma \rangle}.
\]
We can get more from the MGF

**Observation:** If $P$ is an integral polytope in $\mathbb{R}^d_{\geq 0}$, and $F$ is a face of $P$ obtained by setting a collection of variables $\{x_i\}_{i \in \text{ind}}$ to zero, i.e.,

$$F = P \cap \{(x_1, \ldots, x_d) \mid x_i = 0, \forall i \in \text{ind}\},$$

then

$$f(F, z) = f(P, z) \text{ evaluated at } z_i = 0, \forall i \in \text{ind}.$$
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then

$$f(F, z) = f(P, z) \text{ evaluated at } z_i = 0, \forall i \in \text{ind}.$$  

**Example:** Let $P$ be the polytope with vertices $v_1 = (0, 0), v_2 = (2, 0)$ and $v_3 = (0, 2)$.  

\begin{align*}
P &: \quad f(P, z) = 1 + z_1 + z_2 + z_1z_2 + z_2^2. \\
f(F, z) &= 1 + z_2 + z_2^2.
\end{align*}
We can get more from the MGF

**Observation:** If $P$ is an integral polytope in $\mathbb{R}^d_{\geq 0}$, and $F$ is a face of $P$ obtained by setting a collection of variables $\{x_i\}_{i \in \text{ind}}$ to zero, i.e.,

$$F = P \cap \{(x_1, \ldots, x_d) \mid x_i = 0, \forall i \in \text{ind}\},$$

then

$$f(F, z) = f(P, z)$$
 evaluated at $z_i = 0, \forall i \in \text{ind}$.

**Example:** Let $P$ be the polytope with vertices $v_1 = (0, 0), v_2 = (2, 0)$ and $v_3 = (0, 2)$.

$$P : \quad f(P, z) = 1 + z_1 + z_1^2 + z_2 + z_1z_2 + z_2^2.$$  

$$F = P \cap \{x_1 = 0\}, \text{ and } f(F, z) = 1 + z_2 + z_2^2.$$  

$$v_1 = P \cap \{x_1 = 0, x_2 = 0\}, \text{ and } f(v_1, z) = 1.$$
Recall that every face of $B_n$ can be obtained by setting a collection of variables $\{x_{i,j}\}$ to zero. Therefore, for each face $F$ of $B_n$ we can compute the MGF of each dilation $mF$ from $f(mB_n, z)$, and thus we obtain the Ehrhart polynomial of $F$ as well.