# The volume of the Birkhoff polytope

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## Outline

- Introduction to the Birkhoff polytope
- Our formula for the volume of the Birkhoff polytope
- Theories behind the proof

# Birkhoff polytope and its faces

**Definition 1.** The *Birkhoff polytope*, $B_n$, is the set of all *doubly-stochastic matrices*, that is, the real nonnegative matrices with all row and column sums equal to one.

We consider $B_n$ in the $n^2$-dimensional space $\mathbb{R}^{n^2} = \{n \times n \text{ real matrices }\}$.

Faces of $B_n$ :

- The vertices of $B_n$ are the $n \times n$ permutation matrices. The vertex set $V(B_n)$ can be considered as $S_n$, the set of all the permutations on $[n] = \{1, 2, \ldots, n\}$.
- $B_n$ has $n^2$ facets: for each pair of $(i, j)$ with $1 \leq i, j \leq n$, the doubly-stochastic matrices with $(i, j)$th entry equal to $0$ is a facet.
- In general, if $S$ is the union of the nonzero indices of a set of permutation matrices, then the set of doubly-stochastic matrices $M$ with $M(i, j) = 0$ for $(i, j) \notin S$ is a face of $B_n$, and every face arises this way.

## Example: faces of $B_3$

- $$S = \begin{pmatrix} \star & 0 & 0 \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
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## The volume of $B_n$

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Some work on the volume of $B_n$ and its faces:

- Bona gave a lower bound for $\mathrm{Vol}(B_n)$.
- A special face of $B_n$ : the volume of the Chan-Robbins-Yen polytope is a product of Catalan numbers.
- Beck-Pixton calculated $\mathrm{Vol}(B_n)$ up to $n = 10$.

# Our result

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We present closed formulas for all coefficients of the Ehrhart polynomial $i(B_n, m)$.

In particular, the volume of $B_n$ looks like this

$$\mathrm{Vol}(B_n) = \frac{1}{(n-1)^2!} \sum_{\sigma \in S_n = V(B_n)} \sum \frac{\langle c, \sigma \rangle^{(n-1)^2}}{\gamma_1(c,\sigma) \cdots \gamma_{(n-1)^2}(c,\sigma)}.$$

The rational function summands are indexed by combinatorial data.

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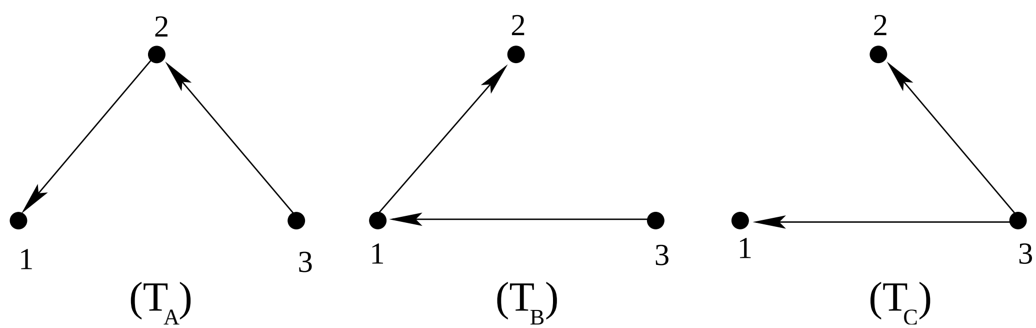

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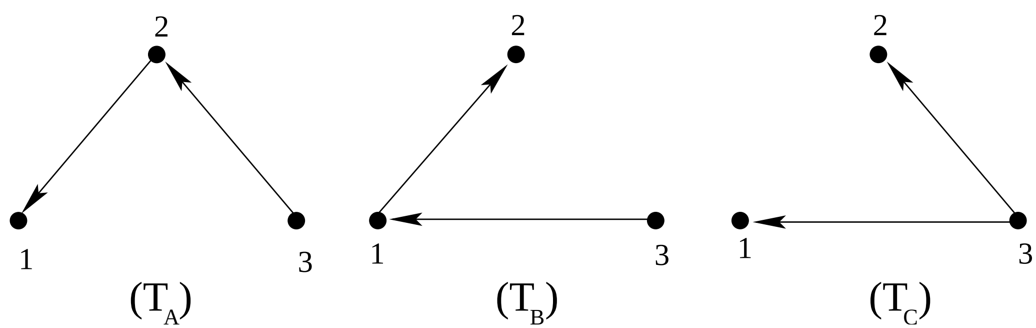


- For any $T \in \mathbf{Arb}(n)$, we denote by $E(T)$ the set of edges of $T$. The cardinality of $E(T)$ is $n-1$. Hence, there are $(n-1)^2$ (directed) edges not in $E(T)$.

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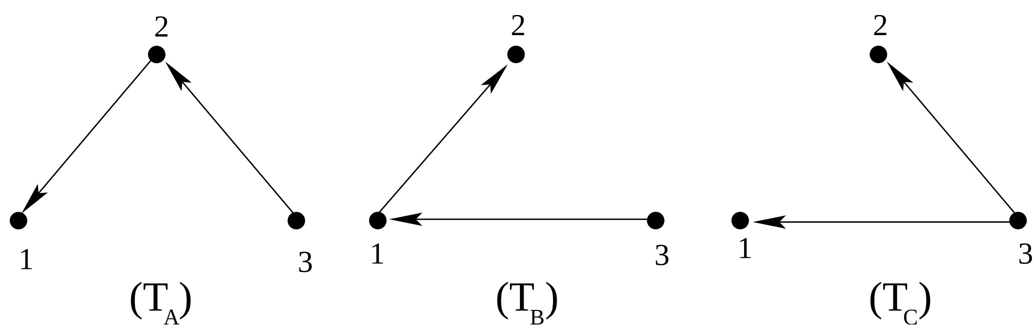
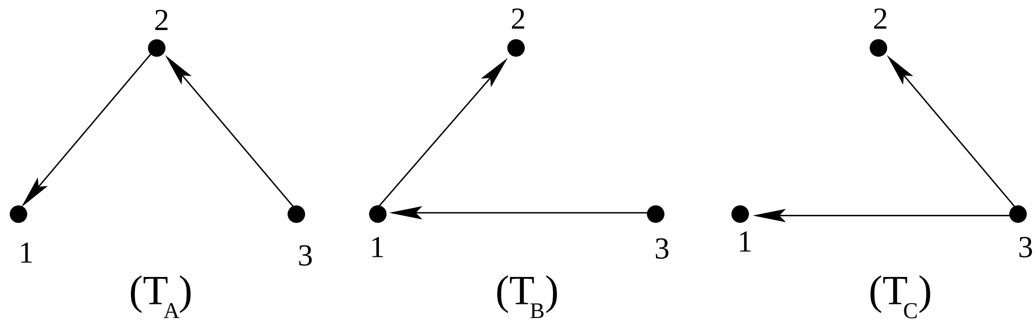



- For any $T \in \mathbf{Arb}(n)$, we denote by $E(T)$ the set of edges of $T$. The cardinality of $E(T)$ is $n-1$. Hence, there are $(n-1)^2$ (directed) edges not in $E(T)$.
- For any $e \notin E(T)$, there is a unique oriented cycle, denoted by $cycle(T+e)$, in $T+e$. We associate a $n \times n$ $(0,-1,1)$-matrix $W^{T,e}$ to it.

# $W^{T,e}$: The matrix associated to $cycle(T+e)$




- The edges in $cycle(T,e)$ all have the same orientation: choose $T_A$ and $e=(1,3)$,

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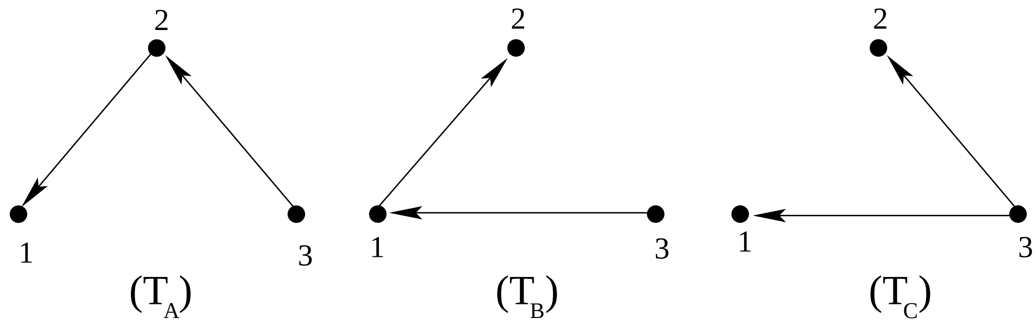


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- There are two vertices $s,t$ in $cycle(T,e)$ such that all the edges are directed from $s$ to $t$ : choose $T_C$ and $e=(2,1)$,

$$W^{T_C,(2,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

## Our main result

**Theorem 2** (DeLoera–L–Yoshida)**.** *The volume of the Birkhoff polytope* $B_n$ *is given by the formula*

$$\mathrm{Vol}(B_n) = \frac{1}{((n-1)^2)!} \sum_{\sigma \in S_n} \sum_{T \in \mathbf{Arb}(n)} \frac{\langle c, \sigma \rangle^{(n-1)^2}}{\prod_{e \notin E(T)} \langle c, W^{T,e} \sigma \rangle}.$$

*In the formula* $c \in \mathbb{R}^{n^2}$ *is* **any** *vector such that* $\langle c, W^{T,e}\sigma \rangle$ *is non-zero for all pairs* $(T, e)$ *of an arborescence* $T$ *and an edge* $e \notin E(T)$ *and all* $\sigma \in S_n$.

*We can for instance choose* $c = (1, 2, 4, 8, \ldots, 2^{n^2-1})$ *to obtain a completely explicit formula.*

# Lattice points

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**Definition 3.** For any polytope $P \subset \mathbb{R}^d$ and some positive integer $m \in \mathbb{N}$, the *$m$th dilated polytope* of $P$ is $mP = \{m\mathbf{x} : \mathbf{x} \in P\}$. We denote by

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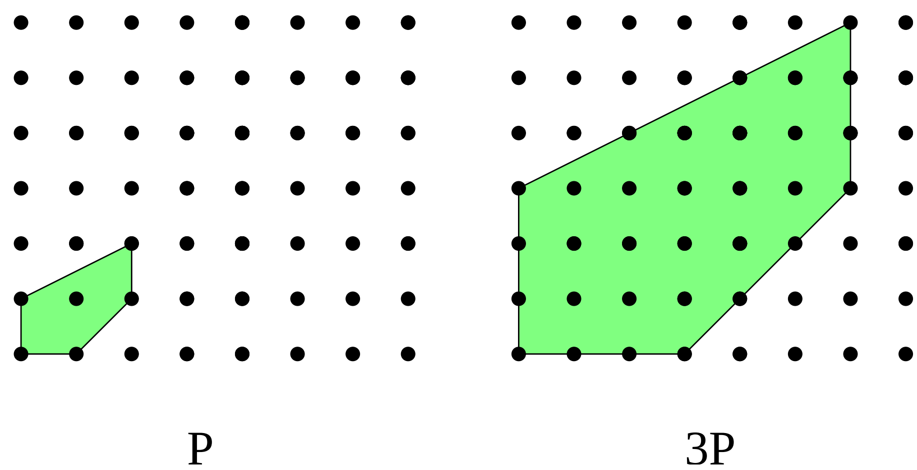
Example:



## Volume/area and lattice points enumerator

**Pick's theorem**: For any integral polygon $Q$:

$$\mathrm{area}(Q) = |Q \cap \mathbb{Z}^2| - \frac{1}{2}|\partial(Q) \cap \mathbb{Z}^2| - 1.$$

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If $P$ is an integral polygon, then so is $mP$.

$$\begin{aligned} i(P,m) &= \text{area}(mP) + \frac{1}{2}|\partial(mP) \cap \mathbb{Z}^d| + 1 \\ &= \text{area}(P)m^2 + \frac{1}{2}|\partial(P) \cap \mathbb{Z}^d|m + 1 \end{aligned}$$

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**Ehrhart's theorem**: Let $P$ be a $d$-dimensional integral polytope, then $i(P, m)$ is a polynomial in $m$ of degree $d$, with the leading coefficient the volume $\text{Vol}(P)$ of $P$.

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of $P$.

## Multivariate generating function

For any polyhedron $P \in \mathbb{R}^d$, we define the *multivariate generating function* (MGF) of $P$ as

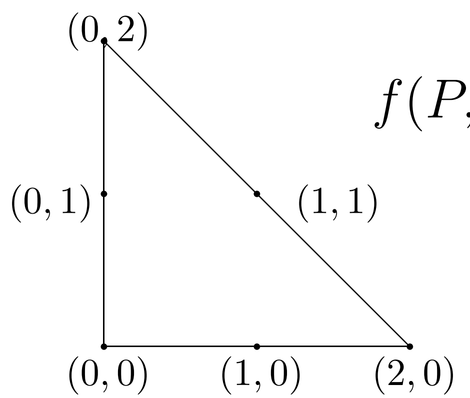
$$f(P, \mathbf{z}) = \sum_{\alpha \in P \cap \mathbb{Z}^d} \mathbf{z}^\alpha,$$

where $\mathbf{z}^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$.

One sees that by setting $\mathbf{z} = (1, 1, \ldots, 1)$, we get the number of lattice points in $P$ if $P$ is a polytope.

Example: Let $P$ be the polytope with vertices $v_1 = (0, 0)$, $v_2 = (2, 0)$ and $v_3 = (0, 2)$.

$P$ :




$$f(P, \mathbf{z}) = z_1^0 z_2^0 + z_1^1 z_2^0 + z_1^2 z_2^0 + z_1^0 z_2^1 + z_1^1 z_2^1 + z_1^0 z_2^2$$
$$= 1 + z_1 + z_1^2 + z_2 + z_1 z_2 + z_2^2.$$

# Why MGF?

It turns out that $f(P, \mathbf{z})$ can be written as a rational function, for any rational polyhedron $P$.

**Lemma 4** (Brion, 1988; Lawrence, 1991)**.** *Let $P$ be a rational polyhedron and let $V(P)$ be the vertex set of $P$. Then, considered as rational funtions,*

$$f(P, \mathbf{z}) = \sum_{v \in V(P)} f(C(P, v), \mathbf{z}),$$

*where $C(P, v)$ is the supporting cone of $P$ at $v$, i.e., the smallest cone with vertex $v$ containing $P$.*

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Remark: We only need to find the MGF for one of the vertices of $B_n$, then apply the action of symmetric group to get the others. We will do this at the vertex associated to the identity permutation matrix, denoted by $I$. We denote by $C_n$ the supporting cone of $B_n$ at $I$.

Our goal is then to find $f(C_n, \mathbf{z})$.

## Unimodular cones

If $K$ is a $d$-dimensional cone in $\mathbb{R}^e$, generated by vectors $\{r_i\}_{1\le i\le d}$ such that the $r_i$'s form a $\mathbb{Z}$-basis of the lattice $\mathrm{span}(\{r_i\}) \cap \mathbb{Z}^e$, then we say $K$ is a *unimodular cone*.

**Lemma 5.** *If $K$ is a $d$-dimensional unimodular cone at an integral vertex $v$ generated by the vectors $\{r_i\}_{1\le i\le d}$, then we have*

$$f(K, \mathbf{z}) = \mathbf{z}^v \prod_{i=1}^{d} \frac{1}{1 - \mathbf{z}^{r_i}}.$$

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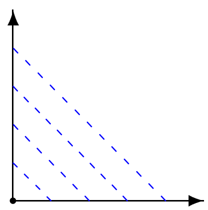

$(0,0)$

A unimodular cone generated by vectors $r_1 = (1,0)$ and $r_2 = (0,1)$.

$f(C(P, v_1), \mathbf{z}) = \mathbf{z}^{(0,0)} \prod_{i=1}^{2} \frac{1}{1-\mathbf{z}^{r_i}} = \frac{1}{(1-z_1)(1-z_2)}$.

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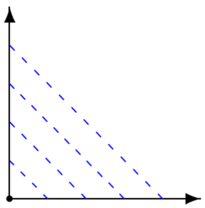


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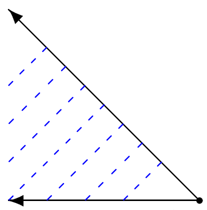


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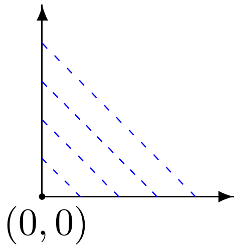


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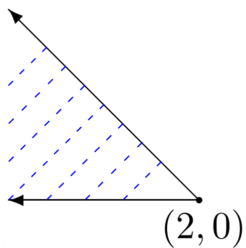


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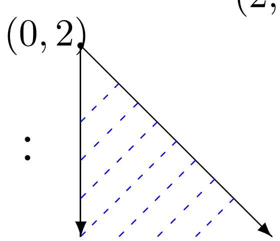


A unimodular cone generated by vectors $r_1 = (0, -1)$ and $r_2 = (1, -1)$.

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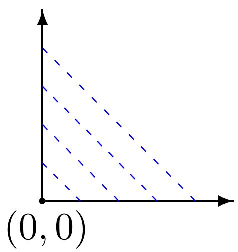


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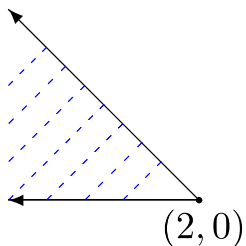


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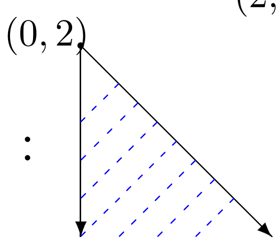


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$$\sum_{i=1}^3 f(C(P,v_i),\mathbf{z}) = \frac{(z_1-z_2)-z_1^4(1-z_2)+z_2^4(1-z_1)}{(1-z_1)(1-z_2)(z_1-z_2)} = 1 + z_1 + z_1^2 + z_2 + z_1 z_2 + z_2^2 = f(P,\mathbf{z}).$$

## Barvinok's algorithm

- Barvinok gave an algorithm to decompose a cone $C$ as a signed sum of simple unimodular cones.
- Using the Brion's polarization trick, we can ignore the lower dimensional cones. This trick involves using the *dual cone* of $C$ instead.

# Barvinok's algorithm

- Barvinok gave an algorithm to decompose a cone $C$ as a signed sum of simple unimodular cones.
- Using the Brion's polarization trick, we can ignore the lower dimensional cones. This trick involves using the *dual cone* of $C$ instead.

**Algorithm:** Input a cone $C$ with vertex $v$

i. Find a dual cone $K$ to $C$.

ii. Apply the Barvinok decomposition to $K$ and get a set of signed unimodular cones $K_i$.

iii. Find dual cone $C_i$ of each $K_i$. (Note $C_i$ is unimodular as well.)

iv. $f(C, \mathbf{z}) = \sum_i \epsilon_i f(C_i, \mathbf{z})$, where $\epsilon_i$ is the sign of $C_i$.

## Apply the algorithm to $C_n$

For step (ii) in the algorithm, we show that any triangulation of the dual cone of $C_n$ gives a set of unimodular cones. Therefore, instead of using Barvinok's method, we use the idea of Gröbner bases of toric ideals to produce triangulations.

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The multivariate generating function of $C_n$ is given by

$$f(C_n, \mathbf{z}) = \sum_{T \in \mathbf{Arb}(\ell, n)} \mathbf{z}^I \prod_{e \notin E(T)} \frac{1}{(1 - \prod \mathbf{z}^{W^{T,e}})},$$

## The MGF of the dilation $mB_n$

We get the multivariate generating function of $B_n$:

$$f(B_n, \mathbf{z}) = \sum_{\sigma \in S_n} \sum_{T \in \mathbf{Arb}(\ell,n)} \mathbf{z}^{\sigma} \prod_{e \notin E(T)} \frac{1}{(1 - \prod \mathbf{z}^{W^{T,e}\sigma})}.$$

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**Theorem 6** (DeLoera–L–Yoshida). *The multivariate generating function of $mB_n$ is given by*

$$f(mB_n, \mathbf{z}) = \sum_{\sigma \in S_n} \sum_{T \in \mathbf{Arb}(\ell,n)} \mathbf{z}^{m\sigma} \prod_{e \notin E(T)} \frac{1}{(1 - \prod \mathbf{z}^{W^{T,e}\sigma})},$$

## From MGF to Ehrhart polynomial and volume

**Corollary 7.** *The Ehrhart polynomial* $i(B_n, m)$ *of* $B_n$ *is given by the formula*

$$i(B_n, m) = \sum_{k=0}^{(n-1)^2} m^k \frac{1}{k!} \sum_{\sigma \in S_n} \sum_{T \in \mathbf{Arb}(\ell,n)} \frac{(\langle c, \sigma \rangle)^k \, \mathrm{td}_{(n-1)^2-k}(\{\langle c, W^{T,e}\sigma \rangle, \ e \notin E(T)\})}{\prod_{e \notin E(T)} \langle c, W^{T,e}\sigma \rangle}.$$

*The symbol* $\mathrm{td}_j(S)$ *is the* $j$*-th Todd polynomial evaluated at the numbers in the set* $S$*. The vector* $c \in \mathbb{R}^{n^2}$ *is any vector such that* $\langle c, W^{T,e}\sigma \rangle$ *is non-zero for all pairs* $(T, e)$ *of an* $\ell$*- arborescence* $T$ *and an arc* $e \notin E(T)$ *and all* $\sigma \in S_n$*.*

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*As a special case, the normalized volume of* $B_n$ *is given by*

$$\operatorname{Vol}(B_n) = \frac{1}{((n-1)^2)!} \sum_{\sigma \in S_n} \sum_{T \in \mathbf{Arb}(\ell, n)} \frac{\langle c, \sigma \rangle^{(n-1)^2}}{\prod_{e \notin E(T)} \langle c, W^{T,e}\sigma \rangle}.$$

## We can get more from the MGF

**Observation:** If $P$ is an integral polytope in $\mathbb{R}^d_{\geq 0}$, and $F$ is a face of $P$ obtained by setting a collection of variables $\{x_i\}_{i \in \mathrm{ind}}$ to zero, i.e.,

$$F = P \cap \{(x_1, \dots, x_d) \mid x_i = 0, \forall i \in \mathrm{ind}\},$$

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Example: Let $P$ be the polytope with vertices $v_1 = (0, 0)$, $v_2 = (2, 0)$ and $v_3 = (0, 2)$.

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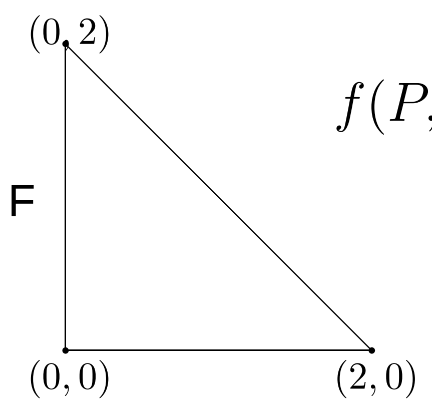


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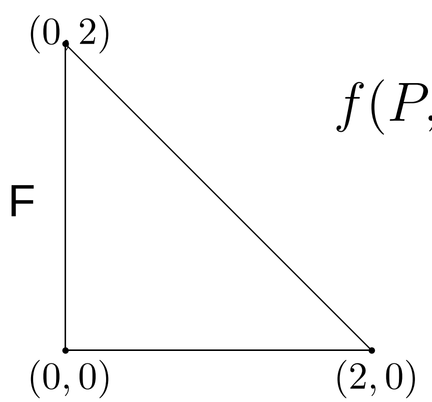


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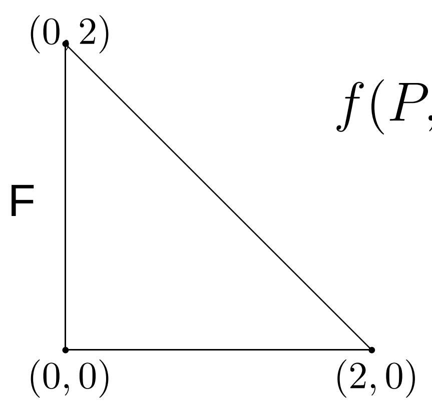


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$v_1 = P \cap \{x_1 = 0, x_2 = 0\}$, and $f(v_1, \mathbf{z}) = 1$.

## Ehrhart polynomials of faces of $B_n$

Recall that every face of $B_n$ can be obtained by setting a collection of variables $\{x_{i,j}\}$ to zero. Therefore, for each face $F$ of $B_n$ we can compute the MGF of each dilation $mF$ from $f(mB_n, \mathbf{z})$, and thus we obtain the Ehrhart polynomial of $F$ as well.