Ehrhart positivity (and McMullen's formula)

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Colloquium

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Outline

- Introduction
  - Polytopes and counting lattice points
  - Ehrhart theory and the Ehrhart positivity question
- Examples of Ehrhart positive families
- McMullen's formula and consequences
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  - A positivity conjecture
- Positivity for generalized permutohedra (joint work with Castillo)
  - Generalized permutohedra
  - Reduction theorem
  - Partial results to the conjecture

PART I:

Introduction

Basic definitions related to polytopes

Definition ($\mathcal{H}$-representation)**.** A *polyhedron* $P \subset \mathbb{R}^d$ is an intersection of finitely many halfspaces:

$$P = \{\mathbf{x} \in \mathbb{R}^d : a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,d}x_d \leq b_i, \ i \in I\},$$

where I is some indexing set, each $a_{i,j} \in \mathbb{R}$, and each $b_i \in \mathbb{R}$.

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The *dimension* of a polytope is the dimension of its affine hull.

Definition. A set F is a *face* of P if there exists $c_0, c_1, \dots, c_d \in \mathbb{R}$ such that

$$c_1 x_1 + c_2 x_2 + \cdots + c_d x_d \leq c_0 \text{ is satisfied for all points } \mathbf{x} \in P$$

and

$$F = P \cap \{\mathbf{x} \in \mathbb{R}^d \; : \; c_1 x_1 + c_2 x_2 + \cdots + c_d x_d = c_0\}.$$

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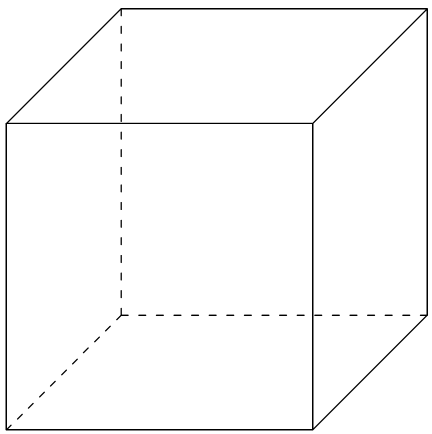

A 3-dimensional cube has:

8 vertices,

12 edges,

6 facets.

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$$P = \mathrm{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

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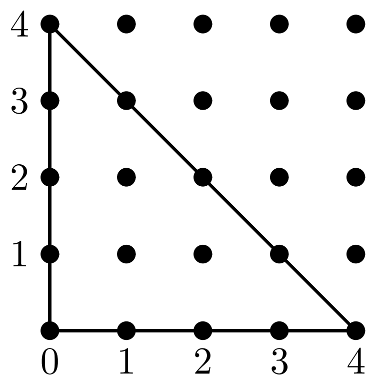

$\mathcal{H}$-representation: $x \geq 0, \ y \geq 0, \ x + y \leq 4.$

$\mathcal{V}$-representation: $\mathrm{conv}\left(\{(0,0), (4,0), (0,4)\}\right)$

Lattice points of a polytope

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Definition. For any polytope $P \subset \mathbb{R}^d$ and positive integer $t \in \mathbb{N}$, the tth dilation of P is $tP = \{t\mathbf{x} : \mathbf{x} \in P\}$. We define

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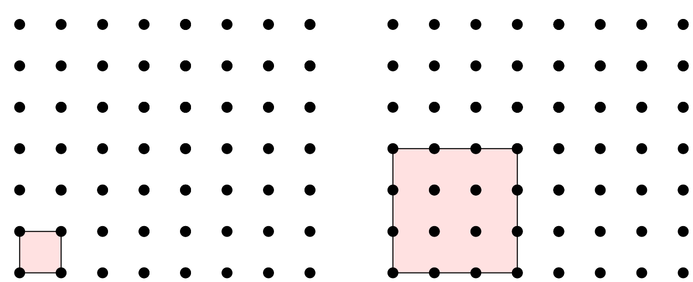
Example: For any d, let $\square_d = \{\mathbf{x} \in \mathbb{R}^d : 0 \le x_i \le 1, \forall i\}$ be the *unit cube* in $\mathbb{R}^d$. Then $t\square_d = \{\mathbf{x} \in \mathbb{R}^d : 0 \le x_i \le t, \forall i\}$ and $i(\square_d, t) = (t+1)^d$.



P 3P

Theorem of Ehrhart (on integral polytopes)

Theorem (Ehrhart)**.** *Let P be a d-dimensional integral polytope. Then $i(P,t)$ is a polynomial in t of degree d.*

Therefore, we call $i(P,t)$ the *Ehrhart polynomial* of P.

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Question.

What about the coefficients of $t^{d-2}, t^{d-3}, \ldots, t^1$ in $i(P, t)$?

We call them the *middle Ehrhart coefficients* of P.

Some negative results

- The *Reeve tetrahedron* T_m is the polytope with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,m)$, where $m \in \mathbb{Z}_{>0}$. Its Ehrhart polynomial is

$$i(T_m, t) = \frac{m}{6}t^3 + t^2 + \frac{12-m}{6}t + 1.$$

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Open Problem: Is it true that any sign pattern is possible?

Ehrhart positivity

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In the literature, different techniques have been used to prove Ehrhart positivity.

PART II:

Examples of Ehrhart positive families

Example I

Polytope: Standard simplex.

Reason: Explicit verification.

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It can be computed that its Ehrhart polynomial is

$$\binom{t+d}{d}.$$

More explicitly, we have

$$\binom{t+d}{d} = \frac{(t+d)(t+d-1)\cdots(t+1)}{d!},$$

which expands positively in powers of t.

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Reason: Roots have negative real parts.

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which is *not* clear if it expands positively in powers of t.

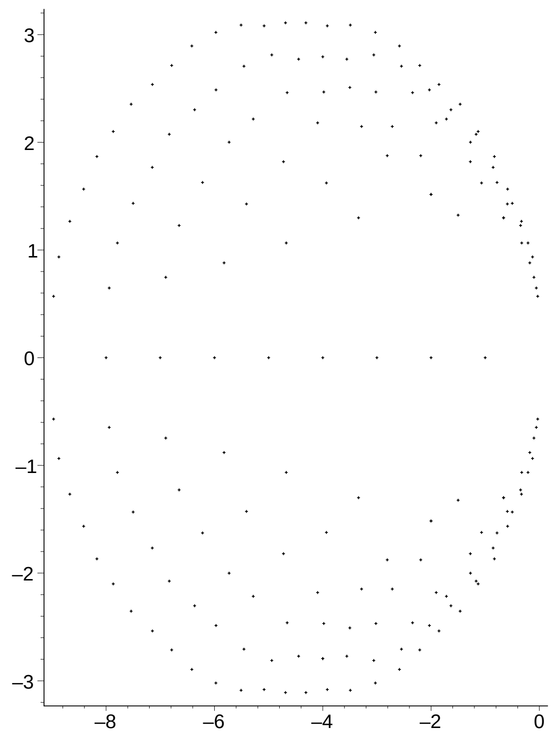
However, according to EC1, Exercise 4.61(b), every root of $i(\Diamond_d, t)$ has real part $-1/2$. Thus it is a product of factors in the form of

$$(t + 1/2) \quad \text{or} \quad (t + 1/2 + ia)(t + 1/2 - ia) = t^2 + t + 1/4 + a^2,$$

where a is real, so Ehrhart positivity follows.

More on roots

The following is the graph (Beck-DeLoera-Pfeifle-Stanley) of roots for the Ehrhart polynomial of the Birkhoff polytope of doubly stochastic $n \times n$ matrices for $n = 2, \ldots, 9$.



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Polytope: Zonotopes.

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Definition. A *zonotope* is the Minkowski sum of a set of line segments. In particular, we let

$$\mathcal{Z}(\mathbf{v}_1, \cdots, \mathbf{v}_k) := [0, \mathbf{v}_1] + [0, \mathbf{v}_2] + \cdots + [0, \mathbf{v}_k].$$

Theorem (Stanley)**.** *The coefficient of t^i in $i(\mathcal{Z}(\mathbf{v}_1, \cdots, \mathbf{v}_k), t)$ is equal to*

$$\sum_{\substack{S=\left\{\mathbf{v}_{j_1}, \dots, \mathbf{v}_{j_i}\right\} \\ \text{linearly independent}}} m(S),$$

where $m(S)$ is the g.c.d. of all $i \times i$ minors of the $d \times i$ matrix

$$M_S = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_{j_1} & \mathbf{v}_{j_2} & \cdots & \mathbf{v}_{j_i} \\ | & | & \cdots & | \end{bmatrix}.$$

The family of zonotopes includes:

- The unit cube $\square_d = \mathcal{Z}(\mathbf{e}_1, \dots, \mathbf{e}_d)$ whose Ehrhart polynomial is

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- The *regular permutohedron*:

$$\begin{aligned} \Pi_{n-1} &= \operatorname{conv}\{(\sigma(1), \sigma(2), \cdots, \sigma(n)) \in \mathbb{R}^n : \sigma \in \mathfrak{S}_n\} \\ &\cong \sum_{1 \le i < j \le n} [e_i, e_j]. \end{aligned}$$

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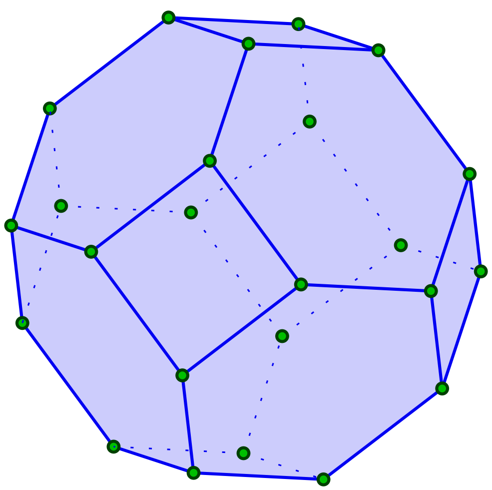

$$i(\Pi_3, t) = 16t^3 + 15t^2 + 6t + 1$$

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For $n > d$, the convex hull of any n distinct points on the moment curve is a *cyclic polytope*.

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Theorem (L.). *If a polytope P satisfies certain higher integrality conditions, the coefficient of t^k in $i(P, t)$ is given by the volume of the projection that forgets the last $d - k$ coordinates.*

Theorem (L.). *For any rational polytope P, there exsits a polytope P' with the same face lattice and Ehrhart positivity.*

Hence,

Ehrhart positivity is **not** a combinatorial property.

Remarks

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However, dilating each coordinate with different parameter works.

Question

Are there other geometric ways to prove Ehrhart positivity?

Other polytopes observed to be Ehrhart positive

- CRY (Chan-Robbins-Yuen).
- Tesler matrices (Mezaros-Morales-Rhoades).
- Birkhoff polytopes. (Beck-DeLoera-Pfeifle-Stanley)
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Littlewood-Richardson

The coefficients in the formula below are the *Littlewood-Richardson coefficients*:

$$s_\lambda \cdot s_\mu = \sum c_{\lambda,\mu}^{\nu} s_\nu.$$

King-Tollu-Toumazet conjectured that the *stretched* Littlewood-Richardson coefficients $c_{t\lambda,t\mu}^{t\nu}$ are polynomials in $\mathbb{N}(t)$.

PART III:

McMullen's formula and consequences

McMullen's formula

Definition. Suppose F is a face of P. The *feasible cone* of P at F, denoted by $\text{fcone}(F, P)$, is the cone of all feasible directions of P at F.

The *pointed feasible cone* of P at F is $\text{fcone}^p(F, P) = \text{fcone}(F, P)/L$, where L is the subspace spanned by F. In general, $\text{fcone}^p(F, P)$ is k-dim'l pointed cone if F is codimensional k.

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In 1975 Danilov asked if it is possible to assign values $\Psi(C)$ to all rational cones C such that the following *McMullen's formula* holds

$$|P \cap \mathbb{Z}^d| = \sum_{F\text{: a face of } P} \alpha(F, P)\,\mathrm{vol}(F).$$

where $\alpha(F, P) := \Psi(\mathrm{fcone}^p(F, P))$.

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McMullen proved it was possible in a non-constructive and nonunique way.

Different Constructions

There are at least three different constructions for Ψ.

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We will use Berline-Vergne's construction, which we will refer to as the *BV-construction*.

A refinement of positivity

Applying McMullen's formula to the dilation tP of P, we obtain

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\begin{aligned}
i(P,t) = |tP \cap \mathbb{Z}^d| &= \sum_{F\text{: a face of } P} \alpha(tF, tP)\,\mathrm{vol}(tF) \\
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$$\sum_{F:\text{ a } k\text{-dimensional face of } P} \alpha(F, P)\,\mathrm{vol}(F).$$

Therefore,

$\alpha(F,P) > 0$ for all k-dim'l face F $\Longrightarrow$ the coefficient of t^k in $i(P,t) > 0$

Moreover,

all α positive $\Longrightarrow$ Ehrhart positive

(BV-)α-positivity

Definition. We say a polytope P is *α-positive* if all the $\alpha(F, P)$ are positive for a given α construction.

We will use *BV-α-positive* for Berline-Vergne's construction.

A refined conjecture

Conjecture. *The regular permutohedron Π_{n-1} is BV-α-positive.*

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Proposition. *The above conjecture implies that all integral generalized permutohedra are Ehrhart positive.*

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A few facts on generalized permutohedra

- A family of polytopes has nice combinatorial properties, first studied by Postnikov.
- Matroid polytopes belong to this family.
- Postnikov showed that a subfamily, called the y-family, has Ehrhart positivity. (Matroid polytopes do not belong to the y-family.)

Ambition

Example V

Polytope: Generalized permutohedra.

Reason: α-positivity.

PART IV:

Positivity for generalized permutohedra

Based on joint work with Castillo.

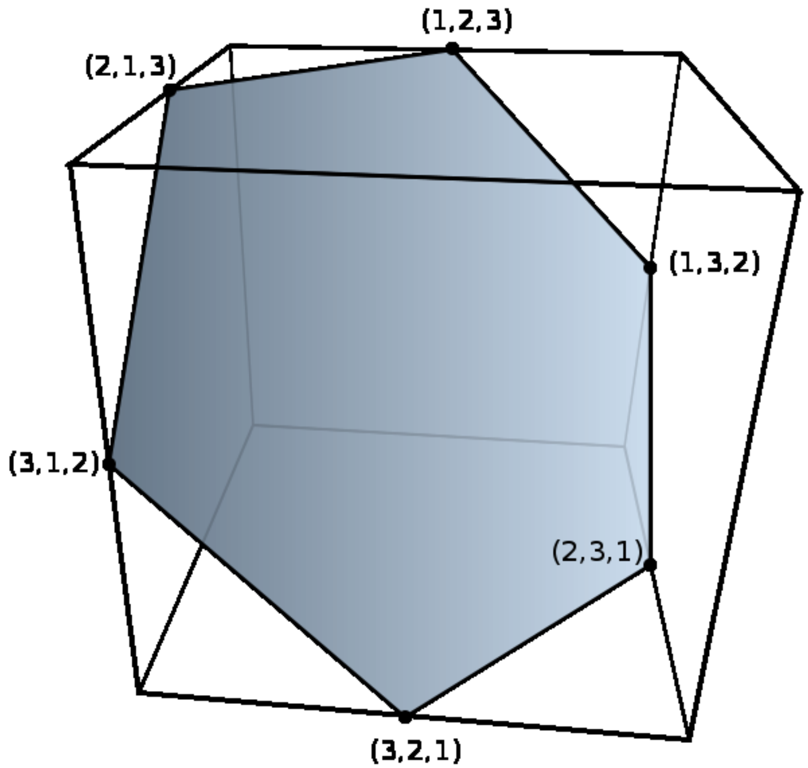
Usual permutohedra

Definition. Suppose $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is a (nondecreasing) sequence. We define the *usual permutohedron*

$$\mathrm{Perm}(\mathbf{v}) := \mathrm{conv}\left\{ \left(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(n)} \right) \ : \ \sigma \in \mathfrak{S}_n \right\}.$$

- If $\mathbf{v} = (1, 2, \cdots, n)$, we get the *regular permutohedron* Π_{n-1}.

Example. Π_2:



Any usual permutohedron in $\mathbb{R}^n$ is $(n-1)$-dimensional.

Generalized permutohedra

Definition (Postnikov)**.** A *generalized permutohedron* is a polytope obtained from a usual permutohedron by moving the facets while keeping the normal directions.

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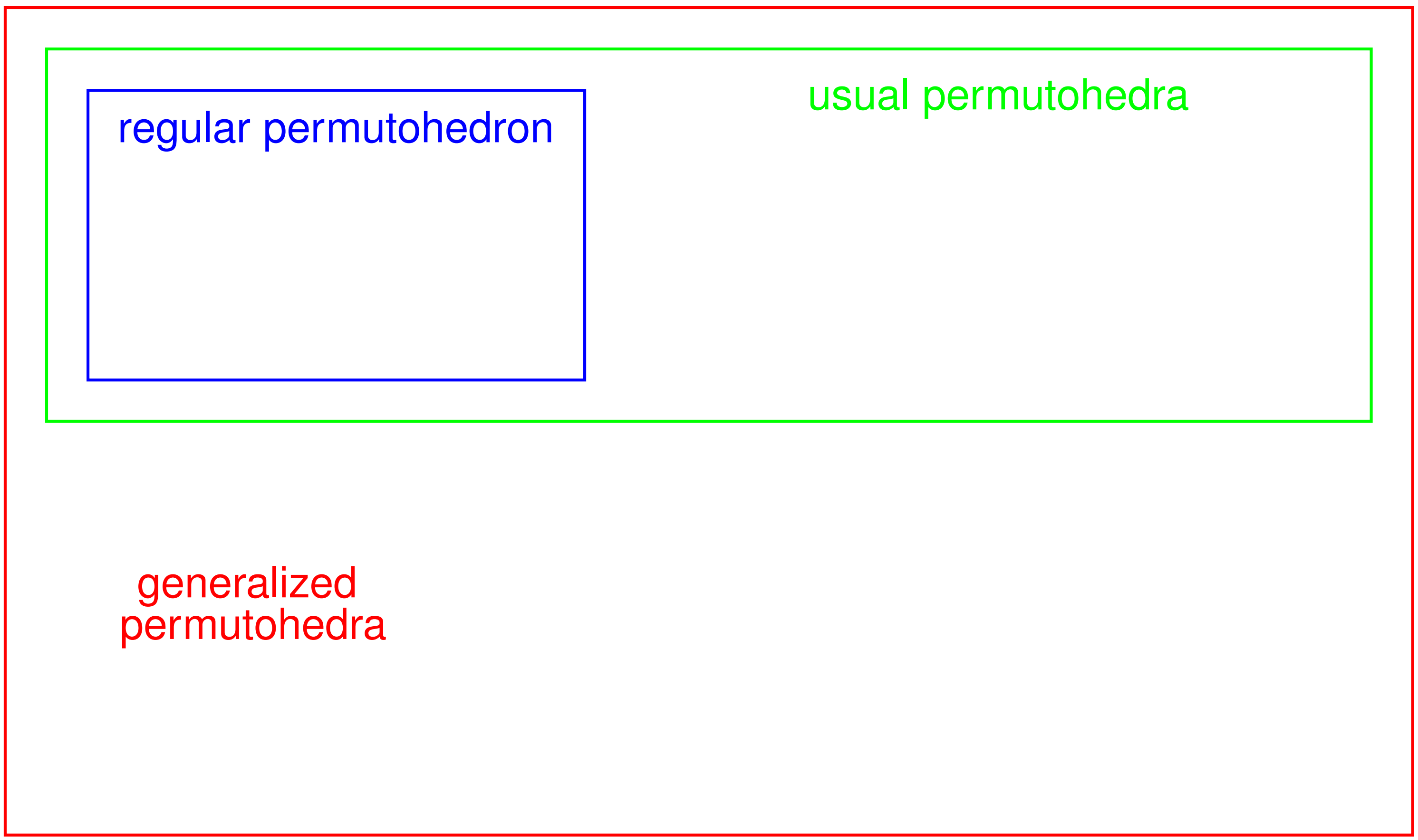

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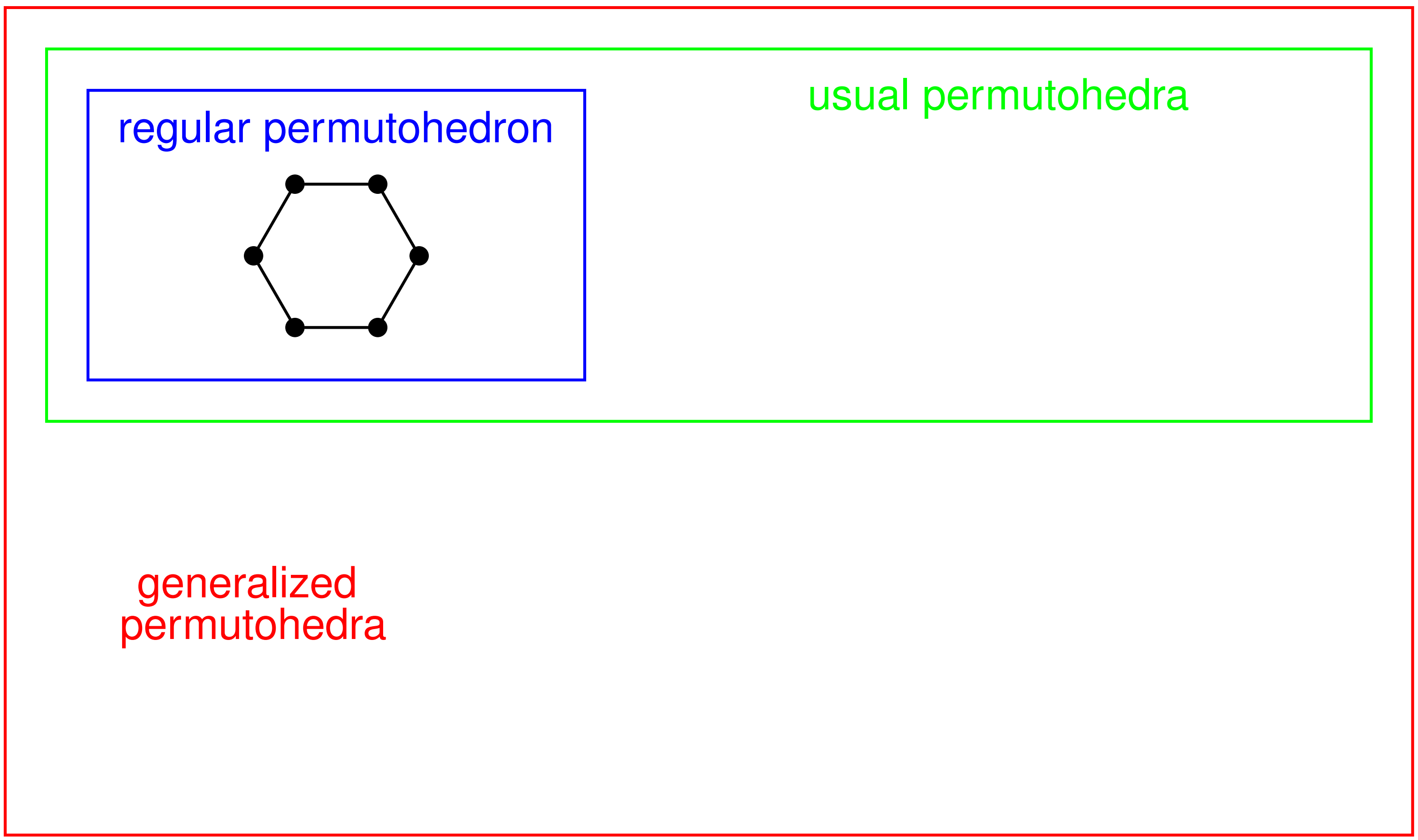

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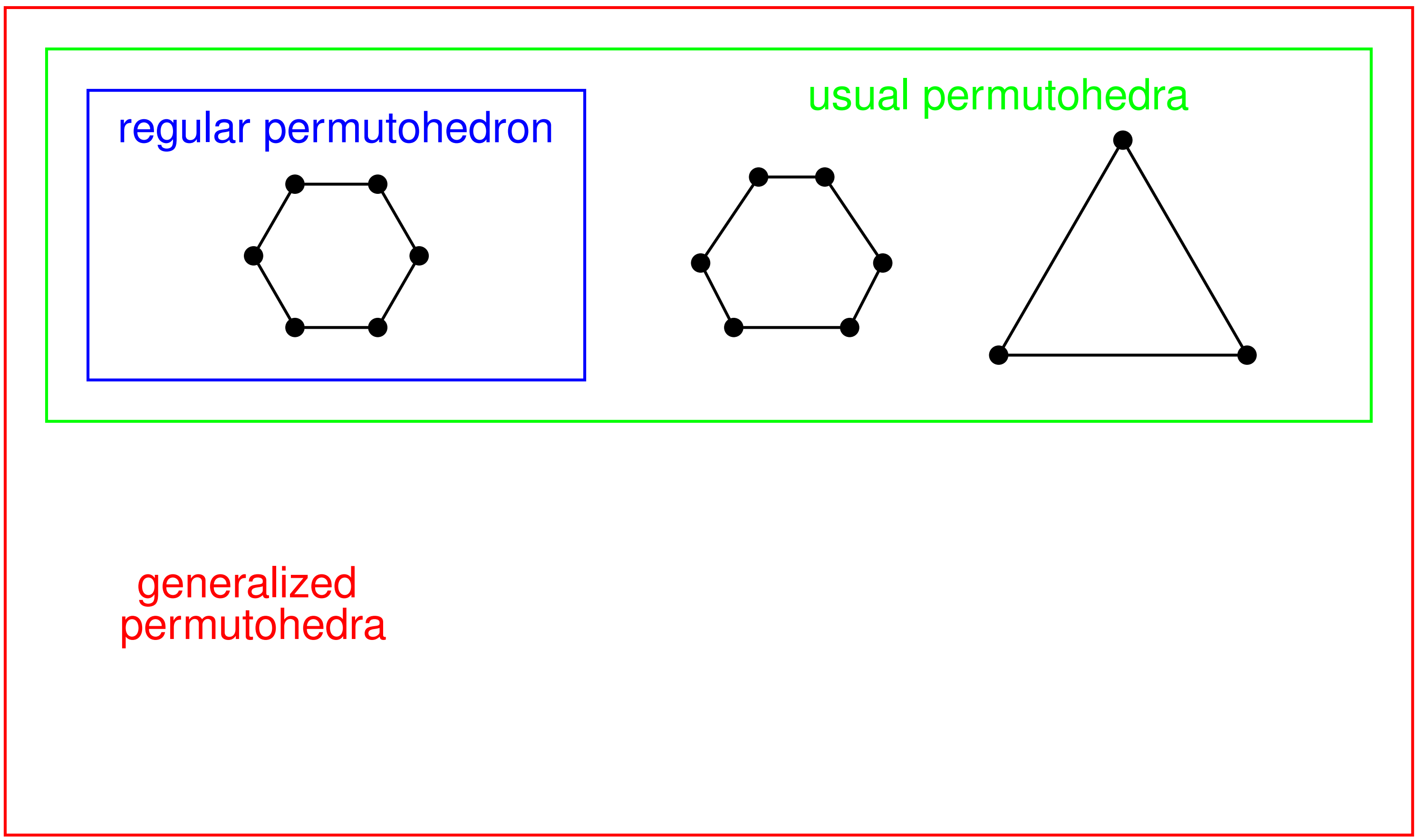

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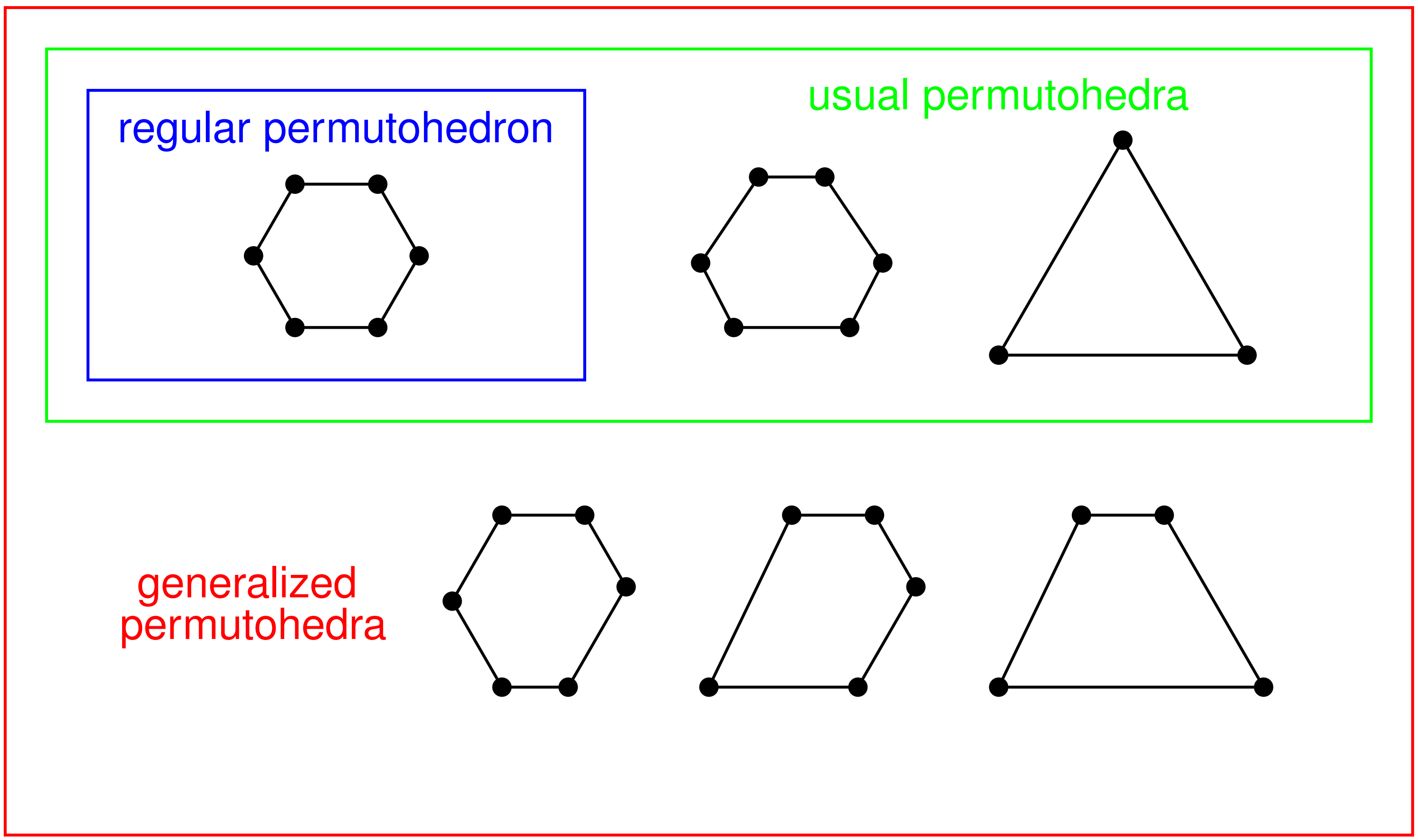

Alternative definition

Let V be the subspace of $\mathbb{R}^n$ defined by $x_1 + x_2 + \cdots + x_n = 0$. The *braid arrangement fan* denoted by B_n, is the complete fan in V given by the hyperplanes

$$x_i - x_j = 0 \quad \text{for all } i \neq j.$$

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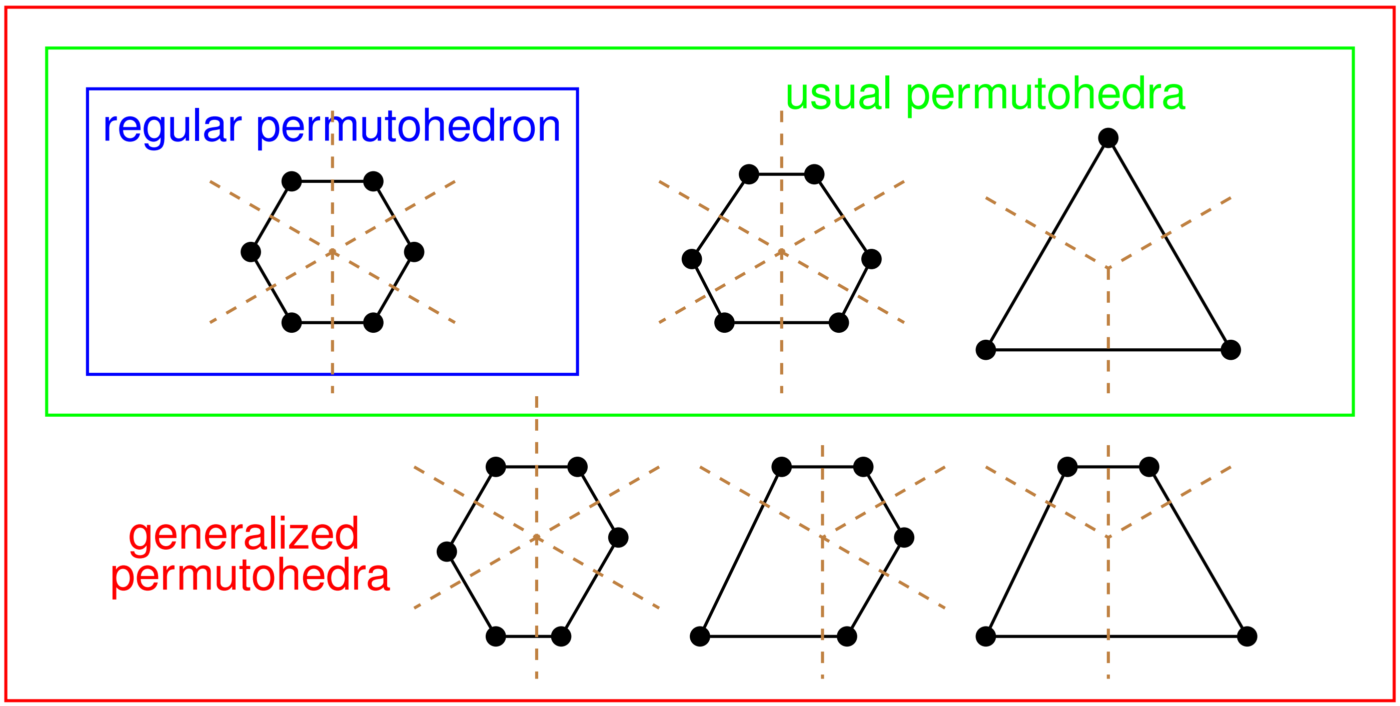

Berline-Vergne's construction

For the rest of this part, we assume that α is the BV-construction.

Important facts about the BV-construction:

- Certain valuation property.
- Invariant under $O_n(\mathbb{Z})$ – orthogonal unimodular transformations, in particular invariant under rearranging coordinates with signs.

Reduction Theorem

Theorem (Castillo-L.)**.** *Suppose* $\alpha(F, \Pi_{n-1}) > 0$ *for any* k*-dimensional face* F *of the regular permutohedron* Π_{n-1}*. Then* $\alpha(G, Q) > 0$ *for any* k*-dimensional face* G *of any generalized permutohedron* Q *in* $\mathbb{R}^n$*.*

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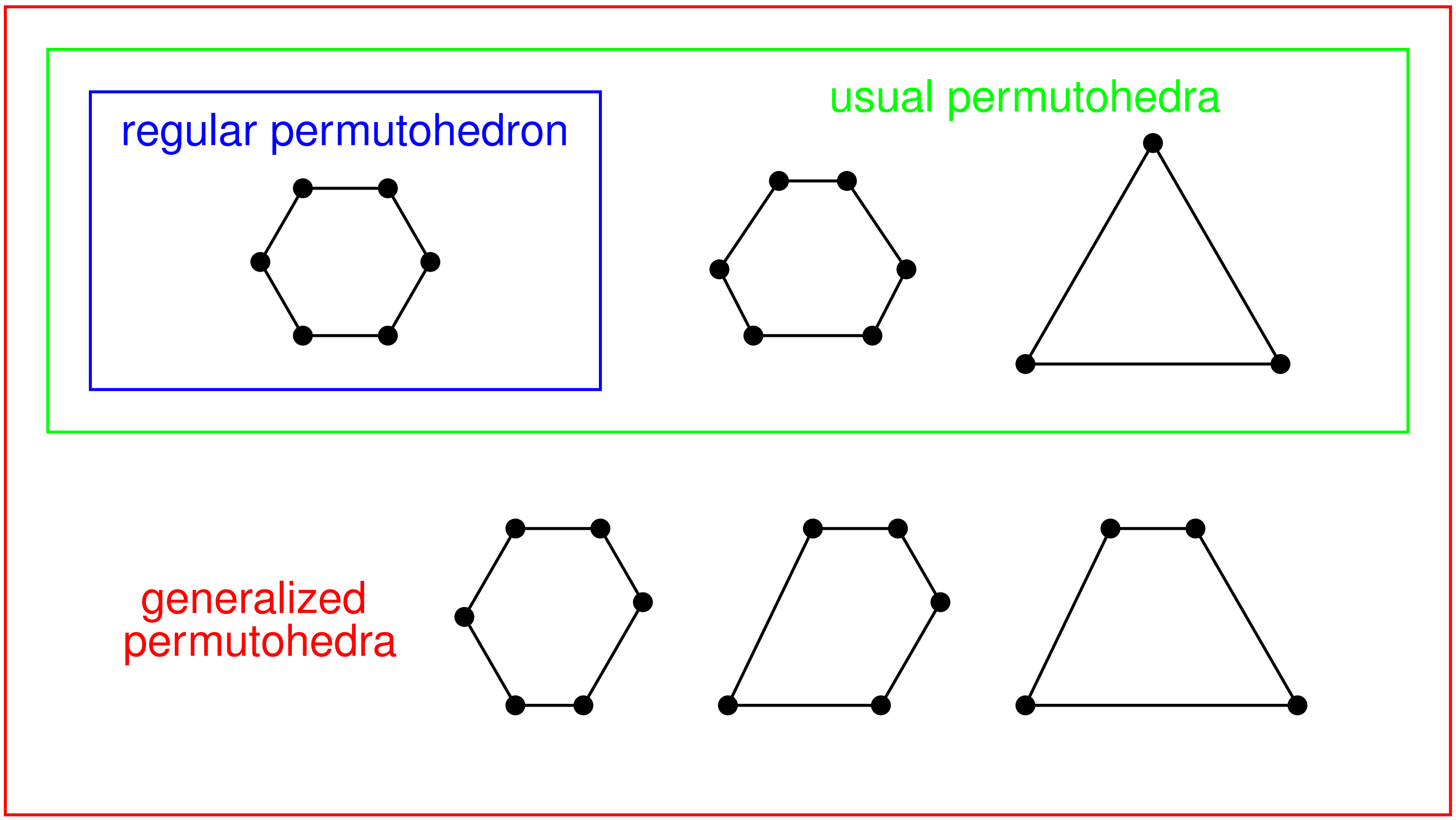

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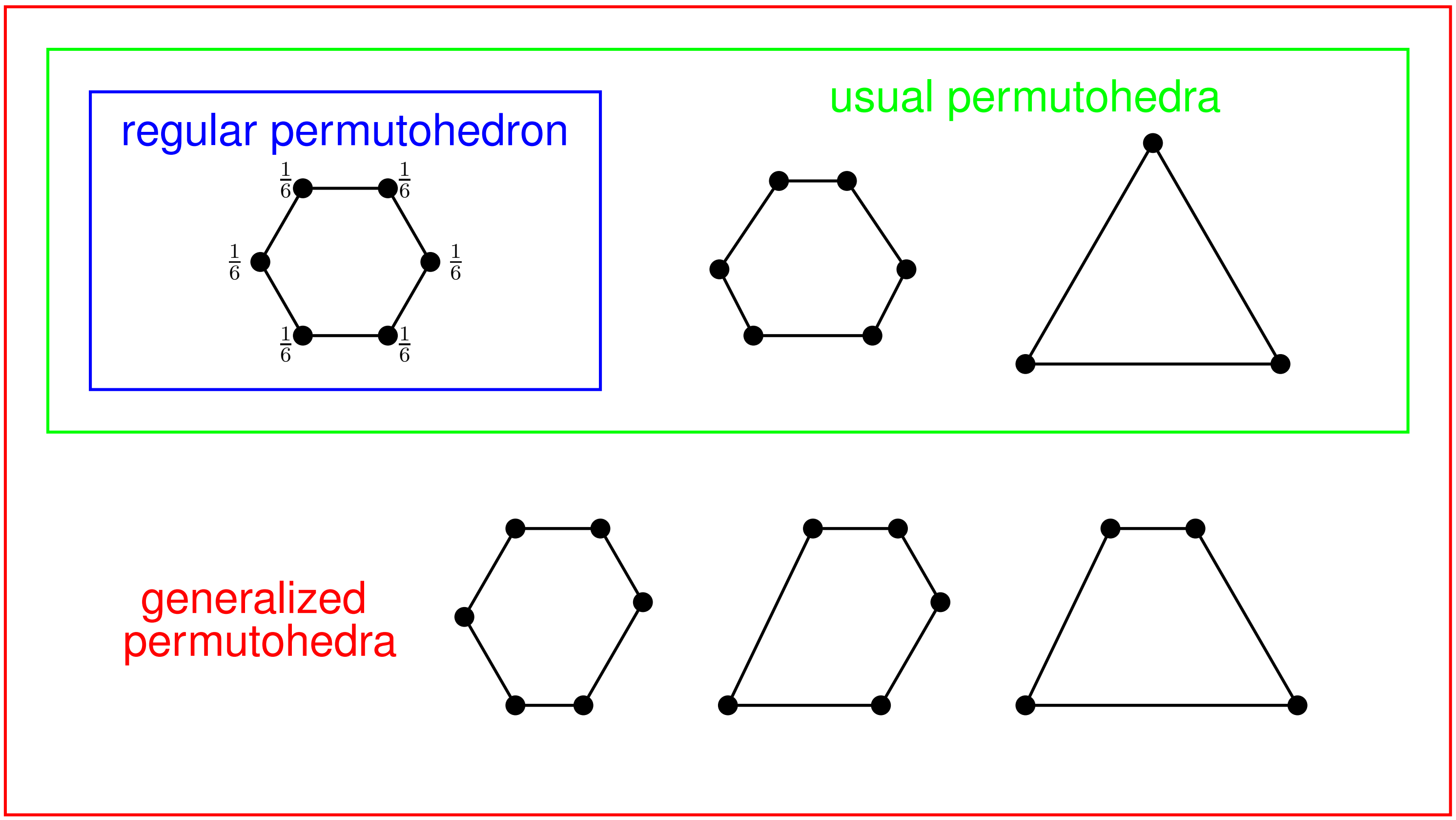

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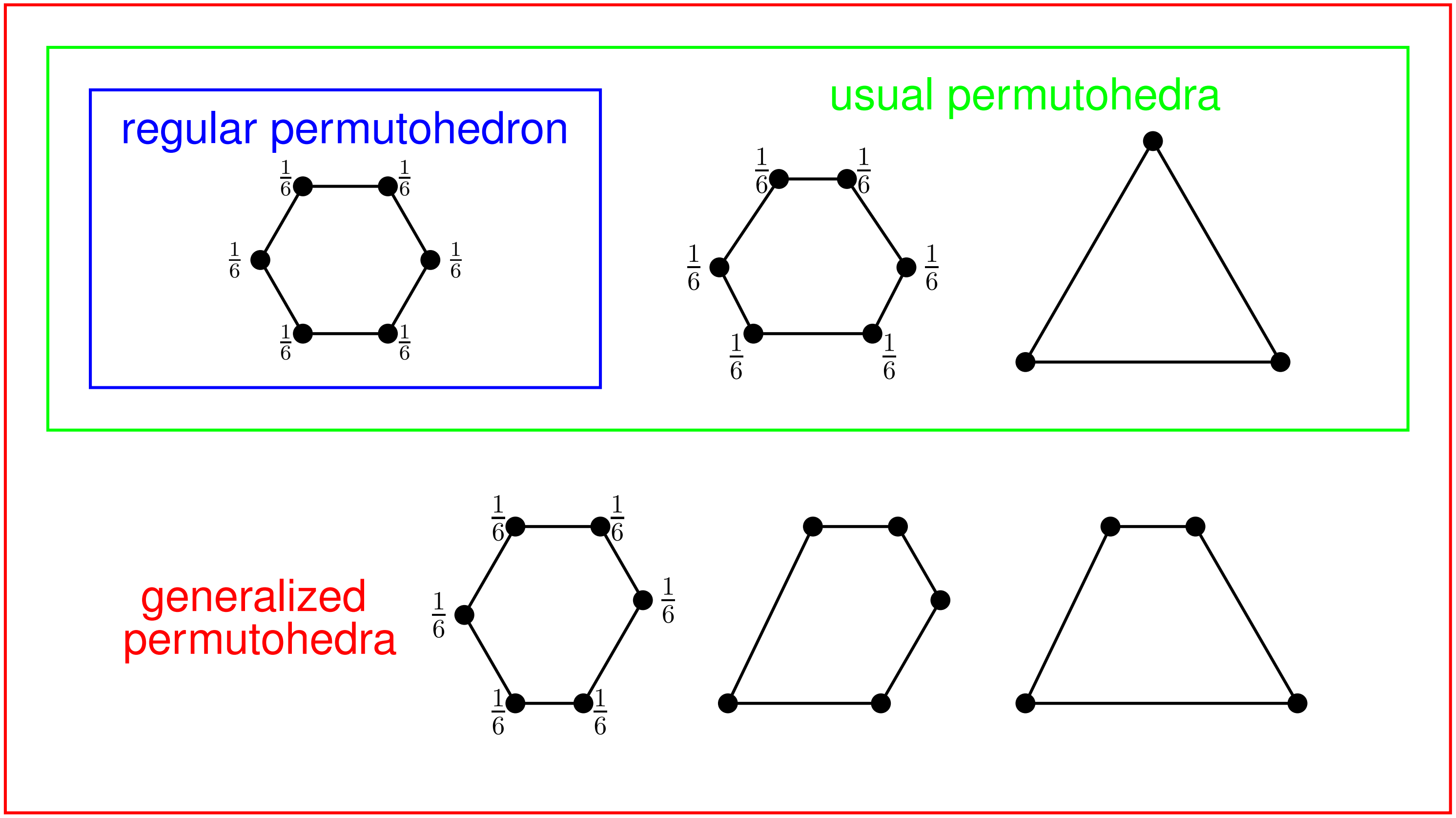

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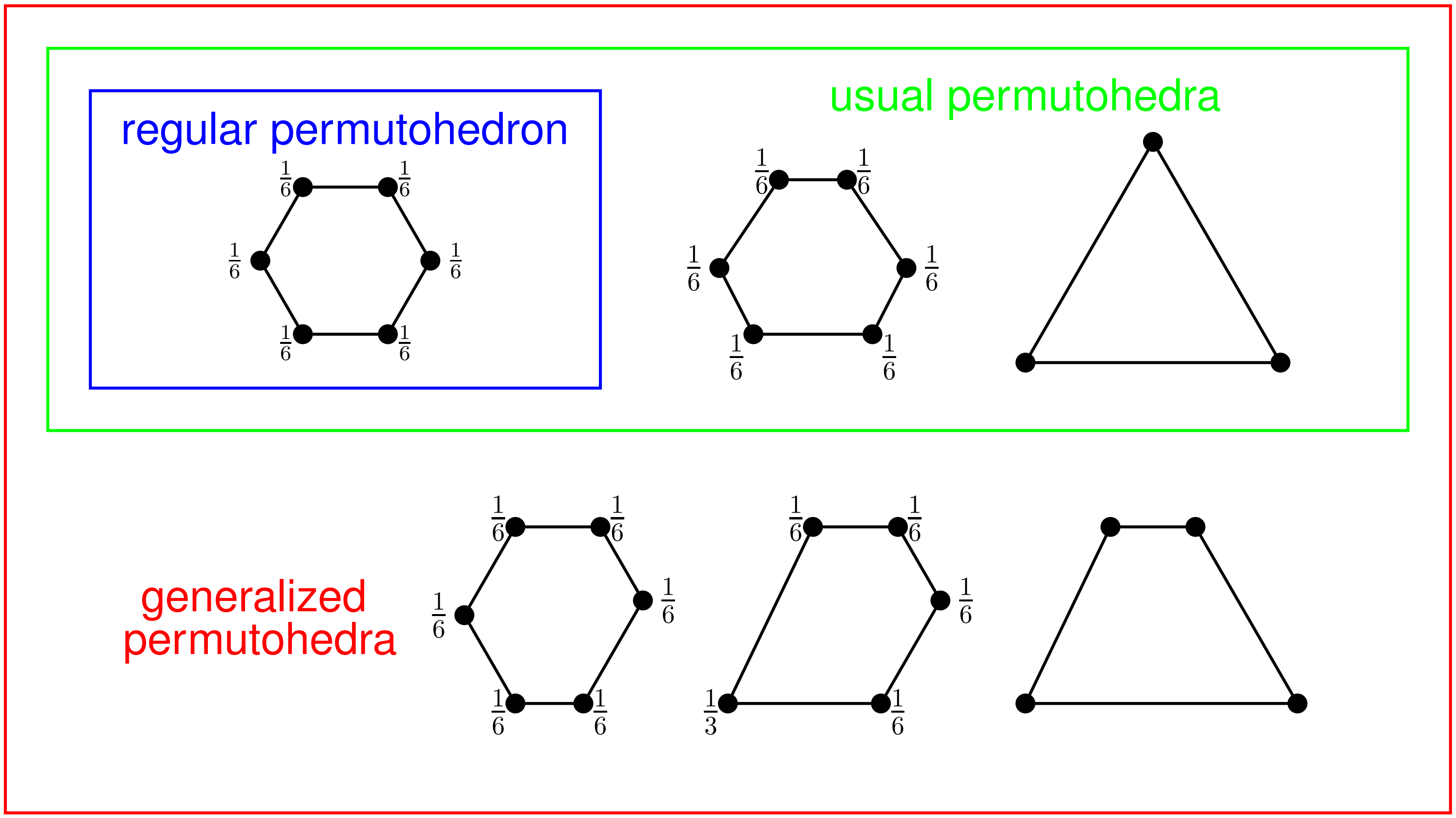

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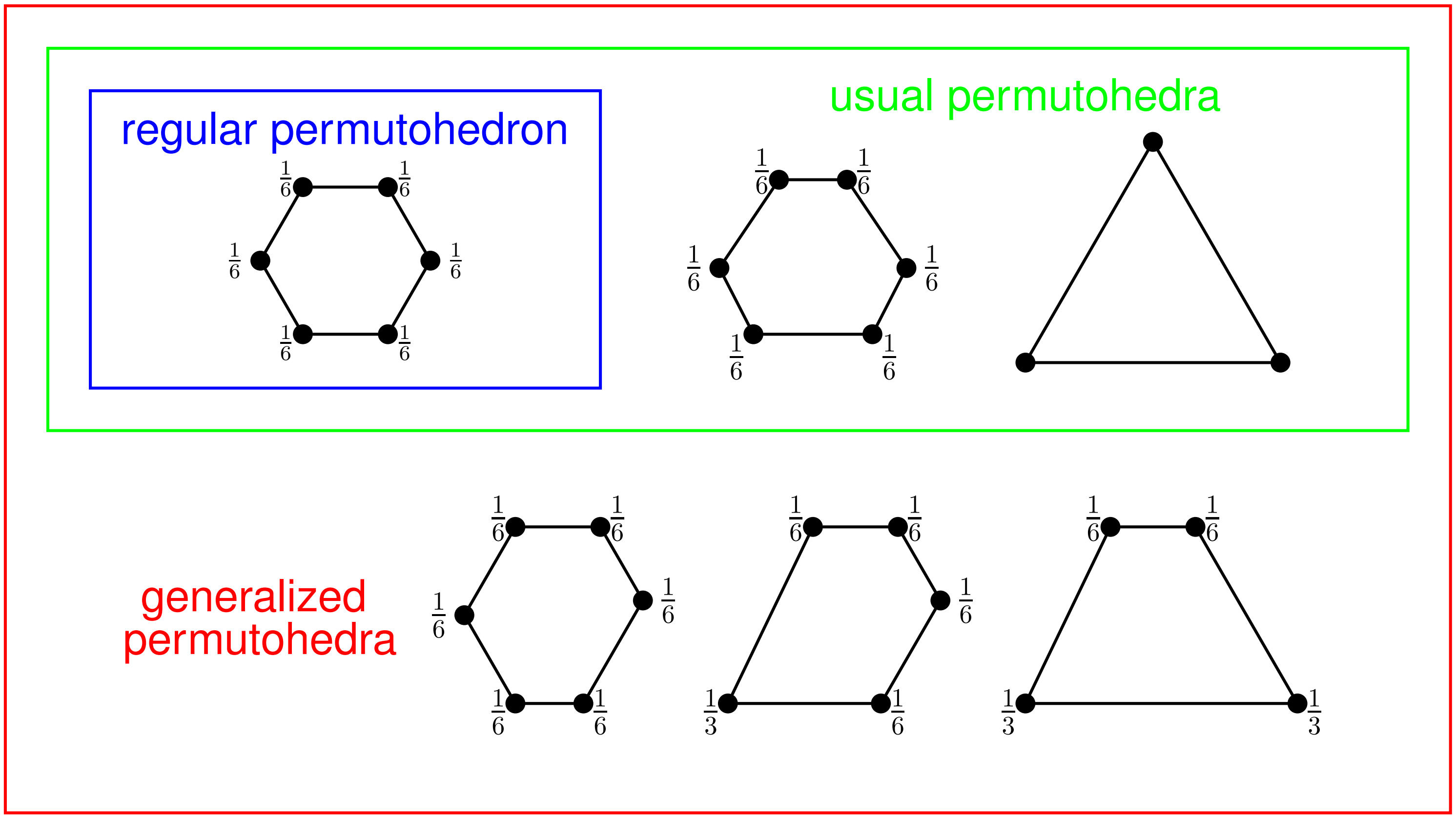

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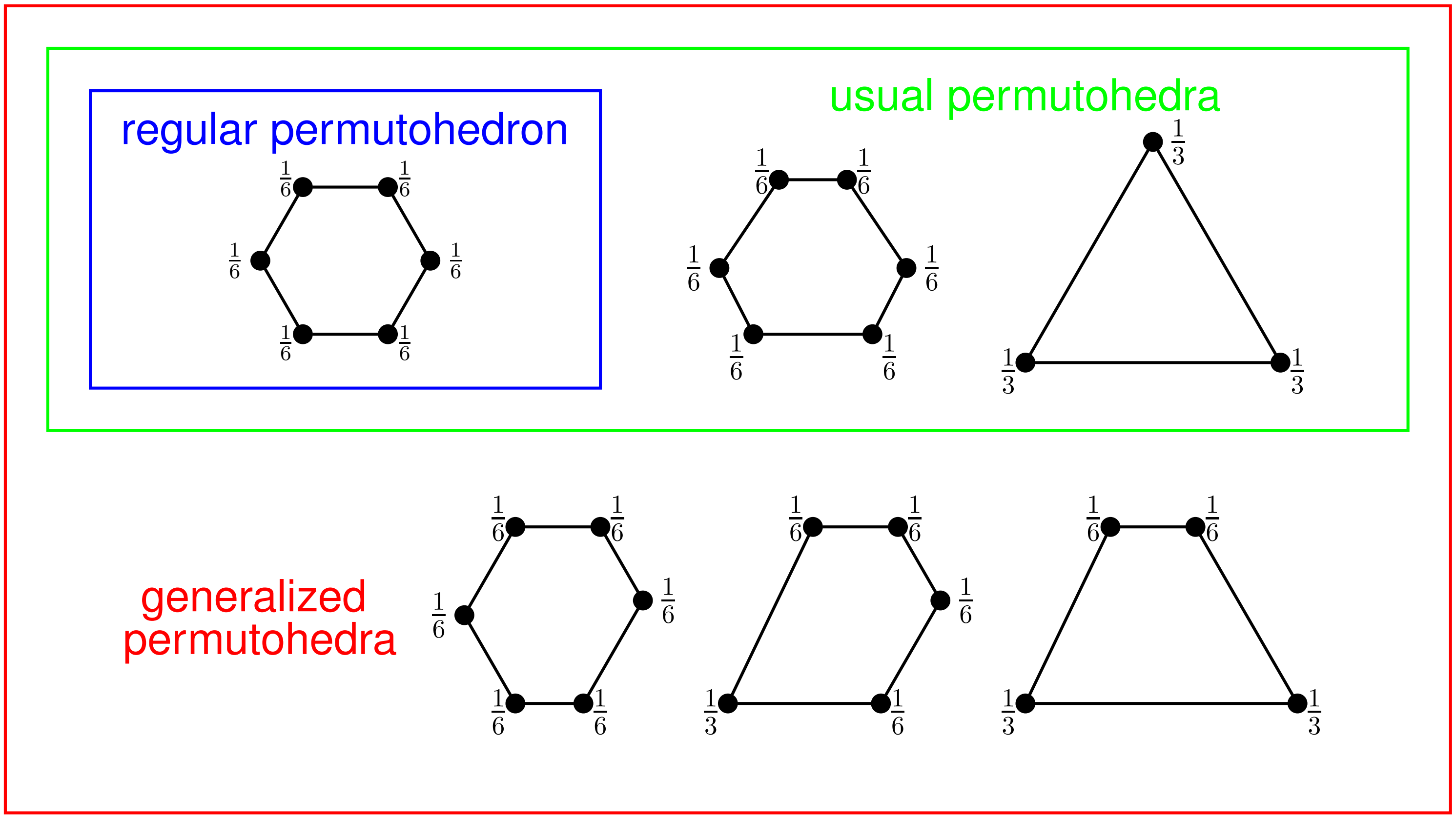

A more general form of the reduction theorem

The reduction theorem is a consequence of the valuation property of the BV-construction for α, thus does not only work for Π_{n-1} and generalized permutohedra.

Theorem (Castillo-L.). *Suppose Q is a deformation of P, or the normal fan of P is a refinement of the normal fan of Q. If $\alpha(F, P) > 0$ for any k-dimensional face F of P, then $\alpha(G, Q) > 0$ for any k-dimensional face G of Q.*

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Applying the reduction theorem, we get:

Corollary (Castillo-L.).

i. *Any integral generalized permutohedron of dimension ≤ 6 is Ehrhart positive.*

ii. *The third and fourth coefficients in the Ehrhart polynomial of any integral generalized permutohedron is positive.*

iii. *The linear coefficient in the Ehrhart polynomial of any integral generalized permutohedron of dimension ≤ 500 is positive.*

Proofs of the first two lemmas

Recall that

$$\alpha(F, P) := \Psi(\mathrm{fcone}^p(F, P)),$$

where Ψ is a function that assigns values to all rational cones.

Fact. 1. Berline-Vergne's Ψ is computed recursively. So lower dimensional cones are easier to compute.

2. If F is a codimension k face of P, then $\mathrm{fcone}^p(F, P)$ is k-dimensional.

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Lemma (Castillo-L.). $\alpha(F, \Pi_{n-1}) > 0$ *for any face F of Π_{n-1} of codimension 2 or 3.*

Proof. We have precise formulas for Ψ of unimodular cones of dimension ≤ 3. Applying these to regular permutohedra, we get α-positivity for faces of codimension ≤ 3. $\square$

The third lemma

Lemma (Castillo-L.). $\alpha(E, \Pi_{n-1}) > 0$ *for any edge* E *of* Π_{n-1} *of dimension* ≤ 500.

The approaches used for the other two lemmas do not work. Since $\alpha(E, \Pi_{n-1})$ is Ψ of an $(n-2)$-dimensional cone, which is very hard to compute directly.

The symmetry property

Lemma. *The valuation* Ψ *(from the BV-construction) is symmetric about the coordinates, i.e., for any cone* $C \in \mathbb{R}^n$ *and any signed permutation* $(\sigma, \mathbf{s}) \in \mathfrak{S}_n \times \{\pm 1\}^n$*, we have*

$$\Psi(C) = \Psi((\sigma, \mathbf{s})(C)),$$

where $(\sigma, \mathbf{s})(C) = \{(s_1 x_{\sigma(1)}, s_2 x_{\sigma(2)}, \dots, s_n x_{\sigma(n)}) \ : \ (x_1, \dots, x_n) \in C\}.$

Idea of the proof of the third lemma

Recall that the coefficient of t^k in $i(P,t)$ is given by

$$\sum_{F\text{: a } k\text{-dimensional face of } P} \alpha(F,P)\,\mathrm{vol}(F).$$

In particular, the coefficient of the linear term is given by

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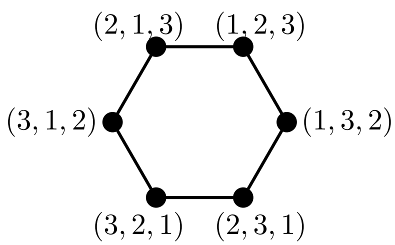
General idea: Suppose you have a family of polytopes such that

- they have same pointed feasible cones (for edges) up to signed permutations, and thus have the same α-values;
- the Ehrhart polynomial of each polytope in the family is known (or at least the linear Ehrhart coefficient is known).

Then as long as you have enough "independent" polytopes in your family, you can figure out the α-values.

Idea of the proof of the third lemma (cont'd)

Example. When $n = 3$: $\Pi_2 = \mathrm{Perm}((1,2,3)) = \mathrm{conv}\{\sigma \ : \ \sigma \in \mathfrak{S}_3\}$.



The pointed feasible cones of the six edges of Π_2 are

$\mathrm{Cone}((1,1,-2))$, $\mathrm{Cone}((2,-1,-1))$, $\mathrm{Cone}((1,-2,1))$,

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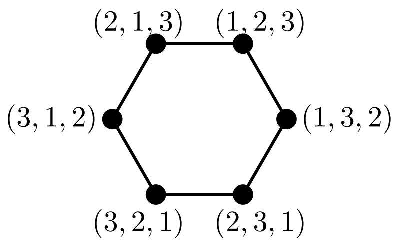

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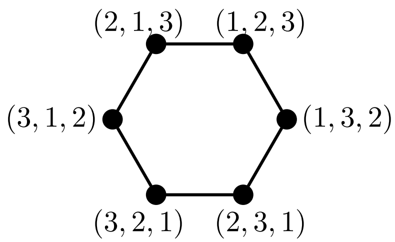

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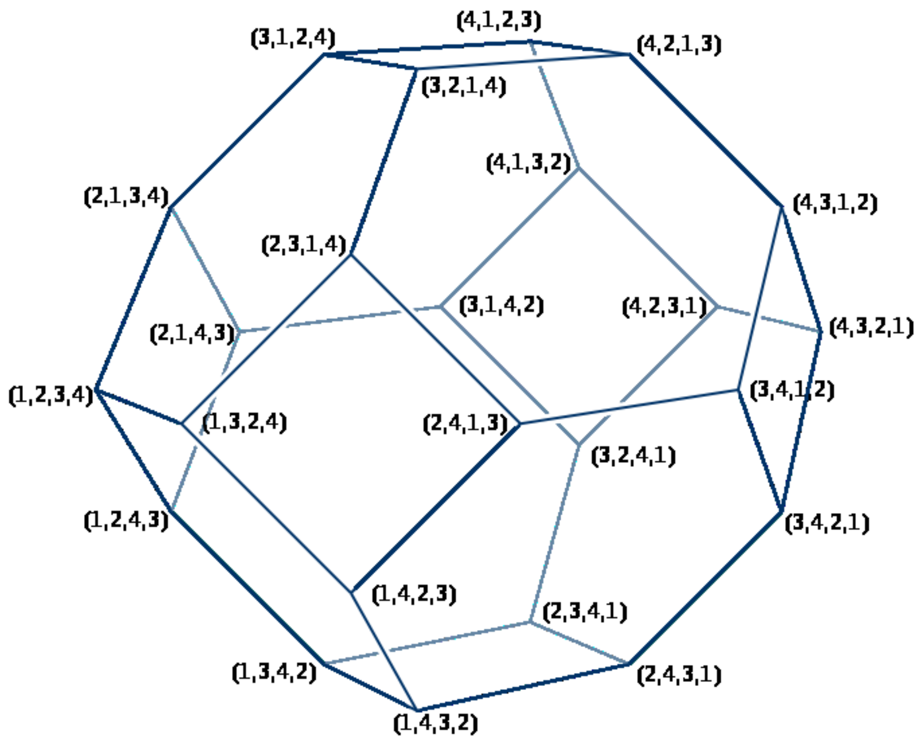
The Ehrhart polynomial of Π_2 is $3t^2 + 3t + 1$. Thus,

$$3 = \sum_E \alpha(E, \Pi_2) \cdot \mathrm{vol}(E) = 6\alpha \quad \Rightarrow \quad \alpha = 1/2 > 0.$$

Idea of the proof of the third lemma (cont'd)

Example. When $n = 4$: $\Pi_3 = \mathrm{Perm}((1,2,3,4)) = \{\sigma \ : \ \sigma \in \mathfrak{S}_4\}$.



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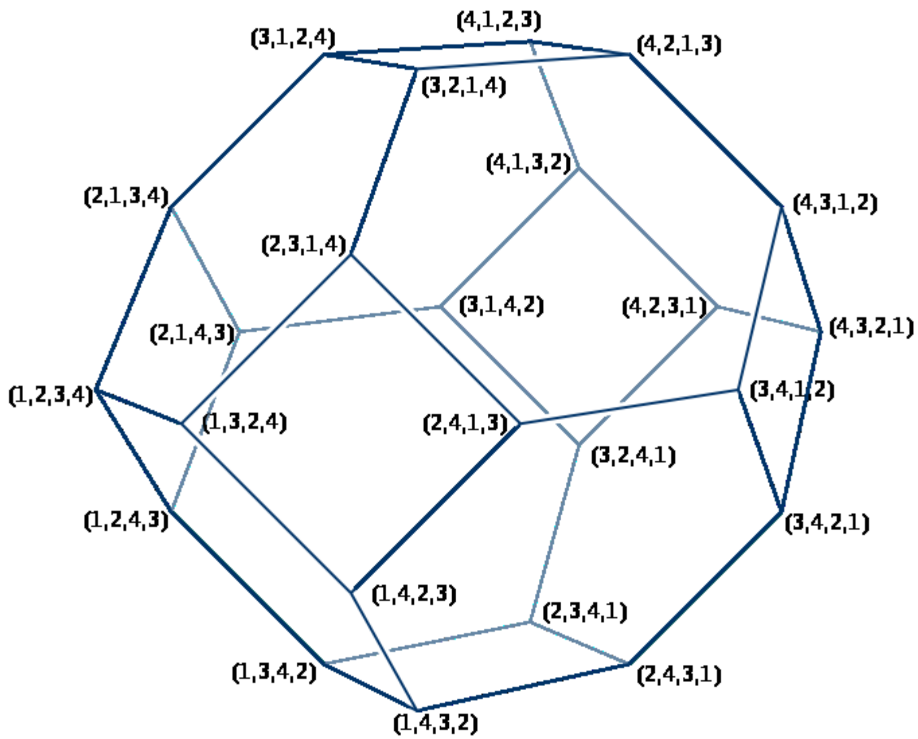

Π_3 have 36 edges of two kinds. 24 short edges have the same α-values, say α_1, and 12 long edges have the same α-values, say α_2.

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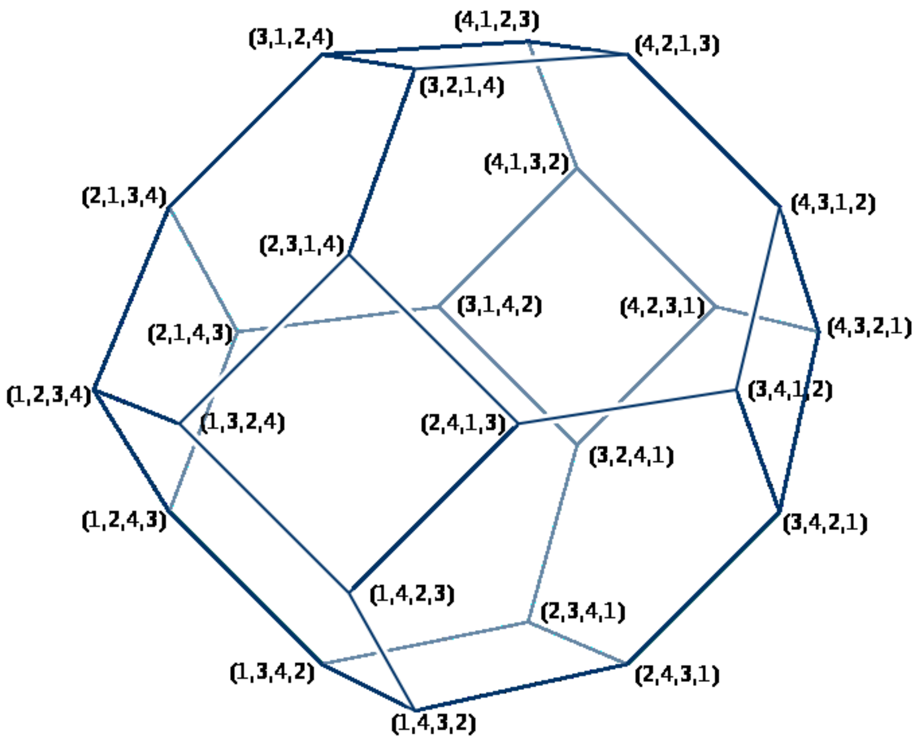

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$$6 = \sum_E \alpha(E, \Pi_3) \cdot \text{vol}(E) = 24\alpha_1 + 12\alpha_2.$$

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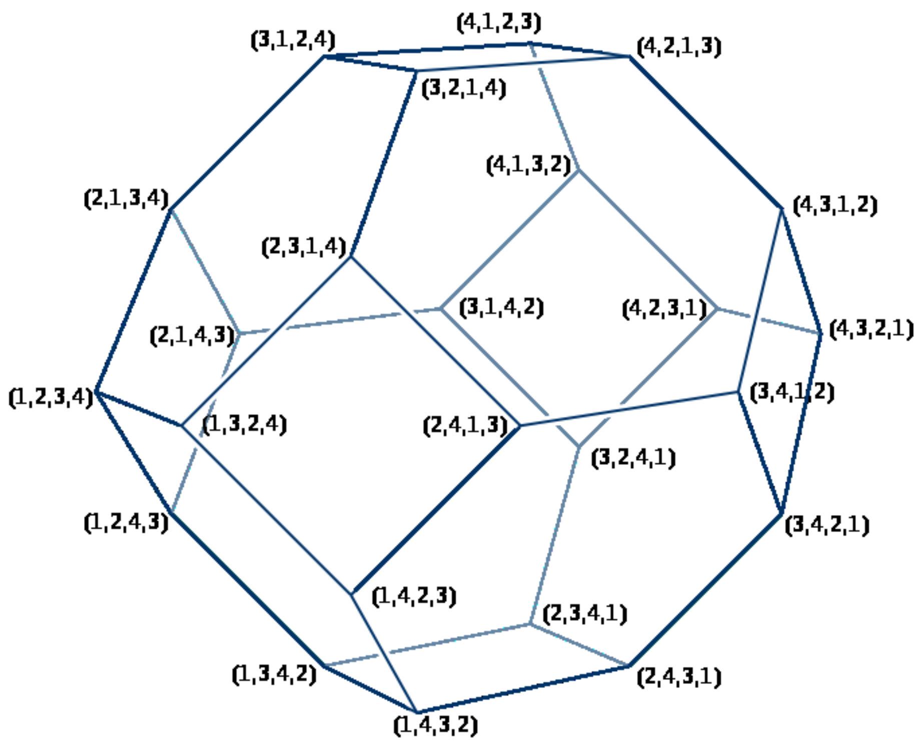

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Not enough equations!

Idea of the proof of the third lemma (cont'd)

Consider the hypersimplex $\Delta_{2,4} = \mathrm{Perm}((0,0,1,1))$. It has 12 edges whose corresponding pointed feasible cones are the same as that of the 12 long edges of Π_3. So they all have α-values α_2.

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$$\frac{7}{3} = \sum_E \alpha(E, \Delta_{2,4}) \cdot \mathrm{vol}(E) = 12\alpha_2.$$

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Therefore, we solve the 2×2 linear system, and get

$$\alpha_1 = \frac{11}{72} > 0, \qquad \alpha_2 = \frac{7}{36} > 0.$$

Idea of the proof of the third lemma (cont'd)

For arbitrary n: The linear Ehrhart coeffcient of some polytopes in the y-family can be easily described. Using these, we were able to set up an explicit triangular linear system for $\{\alpha(E, \Pi_{n-1}) \ : \ E$ is an edge of $\Pi_{n-1}\}$ for any n.

Remark. The number "500" in the lemma can be pushed further.