Volumes and Ehrhart polynomials of polytopes

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Outline

- Preliminaries
- Ehrhart polynomials of cyclic polytopes and lattice-face polytopes
- Further discussion

PART I:

Preliminaries

***Summary*:** We will go over some basic definitions related to polytopes, and then introduce the theory of Ehrhart polynomials.

Basic definitions related to polytopes

Definition 1 ($\mathcal{H}$-representation)**.** A *polyhedron* $P \subset \mathbb{R}^d$ is an intersection of finitely many halfspaces:

$$P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{z}\},$$

for some $A \in \mathbb{R}^{m \times d}, \mathbf{z} \in \mathbb{R}^m$.

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The *dimension* of a polytope is the dimension of its affine hull. A *d-polytope* is a polytope of dimension d in some $\mathbb{R}^e$ $(e \ge d)$.

Definition 2. A *face* of P is any set of the form

$$F = P \cap \{\mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{c}\mathbf{x} = c_0\},$$

where $\mathbf{c}\mathbf{x} \leq c_0$ is satisfied for all points $x \in P$. The *dimension* of a face is the dimension of its affine hull: $\dim(F) := \dim(\mathrm{aff}(F))$.

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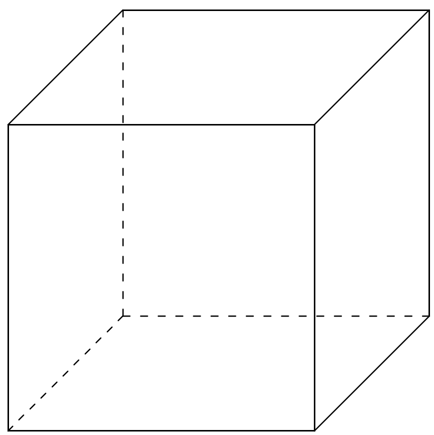
Example:



A 3-dimensional cube has:

8 vertices,

12 edges,

6 facets.

Definition 3 ($\mathcal{V}$-representation)**.** A *(convex) polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \text{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

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Fact 4. The convex hull of all of the vertices of a convex polytope P is P itself:

$$P = \mathrm{conv}(V(P)),$$

where $V(P)$ is the set of vertices of P.

Lattice points

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Definition 5. For any polytope $P \subset \mathbb{R}^d$ and some positive integer $m \in \mathbb{N}$, the *mth dilated polytope* of P is $mP = \{m\mathbf{x} \, : \, \mathbf{x} \in P\}$. We denote by

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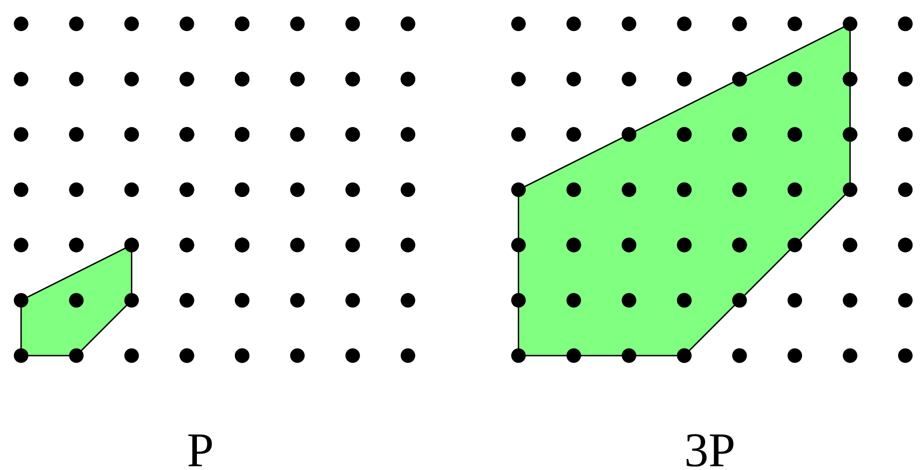

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(ii) When $d = 2$, P is an integral polygon, and so is mP. Pick's theorem states that for any integral polygon Q:

$$\mathrm{area}(Q) = |Q \cap \mathbb{Z}^2| - \frac{1}{2}|\partial(Q) \cap \mathbb{Z}^2| - 1.$$

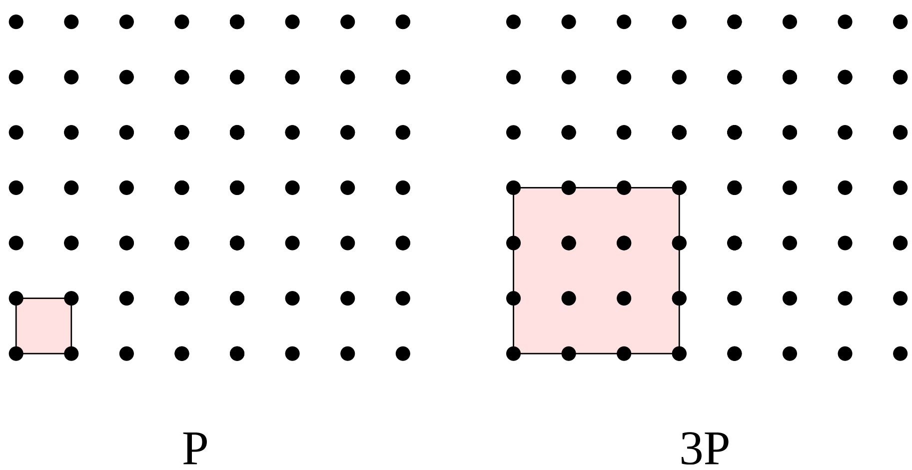
Hence,

$$\begin{aligned} i(P, m) &= \mathrm{area}(mP) + \frac{1}{2}|\partial(mP) \cap \mathbb{Z}^d| + 1 \\ &= \mathrm{area}(P)m^2 + \frac{1}{2}|\partial(P) \cap \mathbb{Z}^d|m + 1 \end{aligned}$$

Examples of integral polytopes

(iii) For any d, let P be the convex hull of the set $\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1\}$, i.e. P is the *unit cube* in $\mathbb{R}^d$. Then it is obvious that

$$i(P, m) = (m+1)^d.$$



$$i(P, 3) = (3+1)^2$$

Theorem of Ehrhart (on integral polytopes)

Theorem 6. *(Ehrhart) Let P be a d-dimensional integral polytope, then $i(P, m)$ is a polynomial in m of degree d.*

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of P.

Example of a rational polytope

When $P = [\frac{1}{3}, \frac{3}{2}]$,

$$i(P,m) = \begin{cases} \frac{7}{6}m + 1, & \text{if } m \equiv 0 \mod 6 \\ \frac{7}{6}m - \frac{1}{6}, & \text{if } m \equiv 1 \mod 6 \\ \frac{7}{6}m + \frac{2}{3}, & \text{if } m \equiv 2 \mod 6 \\ \frac{7}{6}m + \frac{1}{2}, & \text{if } m \equiv 3 \mod 6 \\ \frac{7}{6}m + \frac{1}{3}, & \text{if } m \equiv 4 \mod 6 \\ \frac{7}{6}m + \frac{1}{6}, & \text{if } m \equiv 5 \mod 6 \end{cases}$$

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A function $f : \mathbb{N} \to \mathbb{C}$ (or $f : \mathbb{Z} \to \mathbb{C}$) is a *quasi-polynomial* if there exists an integer $N > 0$ and polynomials $f_0, f_1, \dots, f_{N-1}$ such that

$$f(n) = f_i(n), \text{ if } n \equiv i \mod N.$$

The integer N (which is not unique) will be called a *quasi-period* of f.

Theorem of Ehrhart

Theorem 7. *(Ehrhart) Let P be a d-dimensional rational polytope, then $i(P, m)$ is a quasi-polynomial in m of degree d with quasi-period D, where D is the least common denominator of the vertices of P.*

In particular, when P is an integral polytope, $i(P, m)$ is a polynomial.

Coefficients of Ehrhart polynomials

If P is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, m)$?

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- The constant term of $i(P, m)$ is always 1.
- No results for other coefficients for general polytopes.
  - It is even NOT true that all the coefficients are nonnegative. For example, for the polytope P with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 13)$, its Ehrhart polynomial is

$$i(P, n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$

PART II:

Ehrhart polynomials of cyclic polytopes and lattice-face polytopes

***Summary*:** In this part, we introduce families of polytopes. The coefficients of the Ehrhart polynomials of those polytopes can be described in terms of volumes.

Motivation

Beck, De Loera, Develin, Pfeifle and Stanley conjectured that the Ehrhart polynomial of an integral cyclic polytope has a simple formula.

Recall that given $T = \{t_1, \dots, t_n\}_<$ a linearly ordered set, a d-dimensional *cyclic polytope* $C_d(T) = C_d(t_1, \dots, t_n)$ is the convex hull $\operatorname{conv}\{v_d(t_1), v_d(t_2), \dots, v_d(t_n)\}$ of $n > d$ distinct points $\nu_d(t_i), 1 \le i \le n$, on the moment curve.

The *moment curve* (also known as *rational normal curve*) in $\mathbb{R}^d$ is defined by

$$\nu_d : \mathbb{R} \to \mathbb{R}^d, t \mapsto \nu_d(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix}.$$

Example: $T = \{1, 2, 3, 4\}, d = 3:$

$C_d(T)$ is the convex polytope whose vertices are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}.$

Theorem 8 (L). *For any d-dimensional integral cyclic polytope $C_d(T)$,*

$$i(C_d(T), m) = \mathrm{Vol}(mC_d(T)) + i(C_{d-1}(T), m).$$

Hence,

$$\begin{aligned} i(C_d(T), m) &= \sum_{k=0}^{d} \mathrm{Vol}_k(mC_k(T)) \\ &= \sum_{k=0}^{d} \mathrm{Vol}_k(C_k(T)) m^k, \end{aligned}$$

where $\mathrm{Vol}_k(mC_k(T))$ is the volume of $mC_k(T)$ in k-dimensional space, and by convention we let $\mathrm{Vol}_0(mC_0(T)) = 1$.

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- $C_{d-3}(T) = \mathbb{R}^0 : i(C_{d-3}(T), m) = 1.$
- $2, 4, 3$ and 1 are the volumes of $C_3(T)$, $C_2(T)$, $C_1(T)$ and $C_0(T)$, respectively.

If we define $\pi^k : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-k}$ to be the map which forgets the last k coordinates of a point, then $\pi^k(C_d(T)) = C_{d-k}(T)$. So when $P = C_d(T)$ is an integral cyclic polytope, we have that

$$i(P, m) = \mathrm{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{d-k}(P))m^k, \tag{9}$$

where $\mathrm{Vol}_k(P)$ is the volume of P in k-dimensional Euclidean space $\mathbb{R}^k$.

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where $\mathrm{Vol}_k(P)$ is the volume of P in k-dimensional Euclidean space $\mathbb{R}^k$.

Question: Are there other integral polytopes which have the same form of Ehrhart polynomials as cyclic polytopes?

In other words, what kind of integral d-polytopes P are there whose Ehrhart polynomials will be in the form of (9)?

Properties of integral cyclic polytopes

What are some key properties of an integral cyclic polytope $C_d(T)$?

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For $d \geq 2$, for any d-subset $T' \subset T$, let $U = \nu_d(T')$ be the corresponding subset of the vertex set $V = \nu_d(T)$ of $C_d(T)$. Then:

a) $\pi(\mathrm{conv}(U)) = \pi(C_d(T')) = C_{d-1}(T')$ is an integral cyclic polytope, and

b) $\pi(\mathrm{aff}(U) \cap \mathbb{Z}^d) = \mathbb{Z}^{d-1}$. In other words, after dropping the last coordinate of the lattice of $\mathrm{aff}(U)$, we get the $(d-1)$-dimensional lattice.

Example of condition b): $\pi(\mathrm{aff}(U) \cap \mathbb{Z}^d) = \mathbb{Z}^{d-1}$

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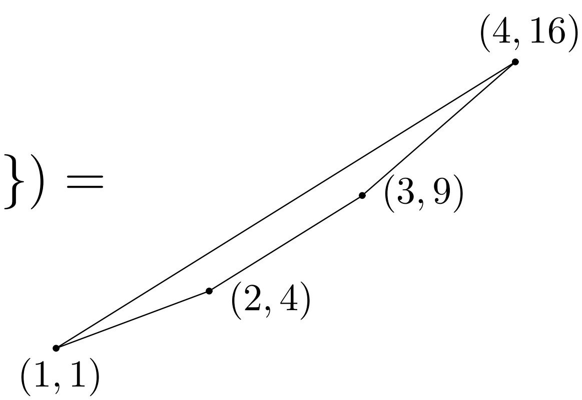
Example: $T = \{1, 2, 3, 4\}, d = 2, T' = \{1, 3\}, U = \{(1, 1), (3, 9)\}.$

$P = C_2(\{1, 2, 3, 4\}) =$



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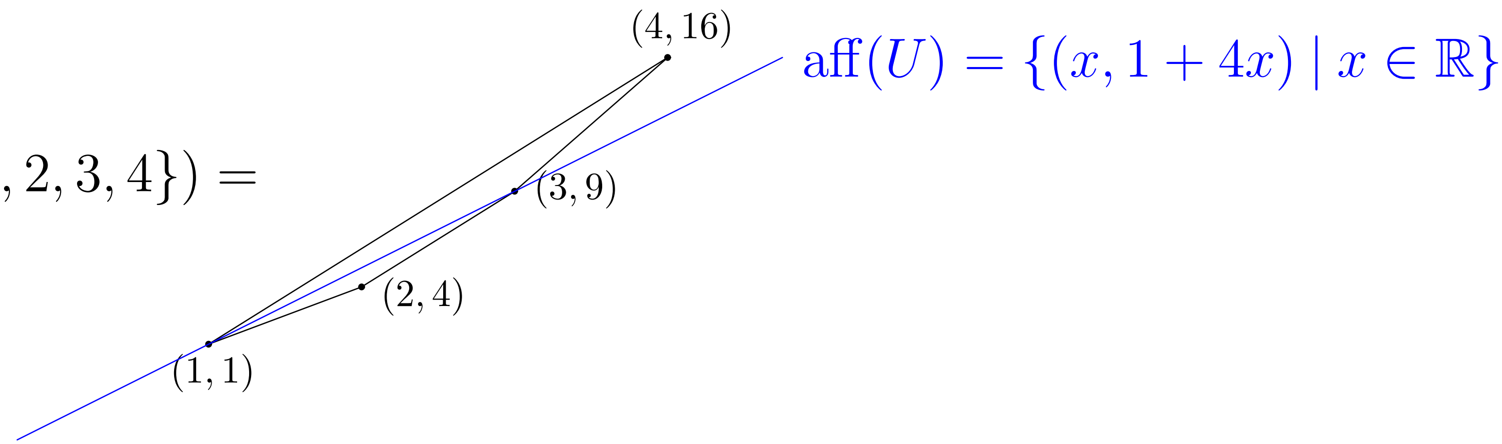

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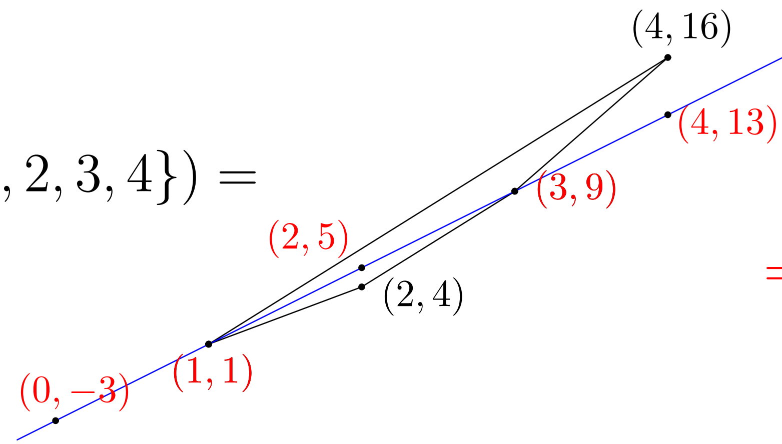

$\mathrm{aff}(U) = \{(x, 1 + 4x) \mid x \in \mathbb{R}\}$

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$= \{\cdots, (0, -3), (1, 1), (2, 5), (3, 9), (4, 13), \cdots\}$

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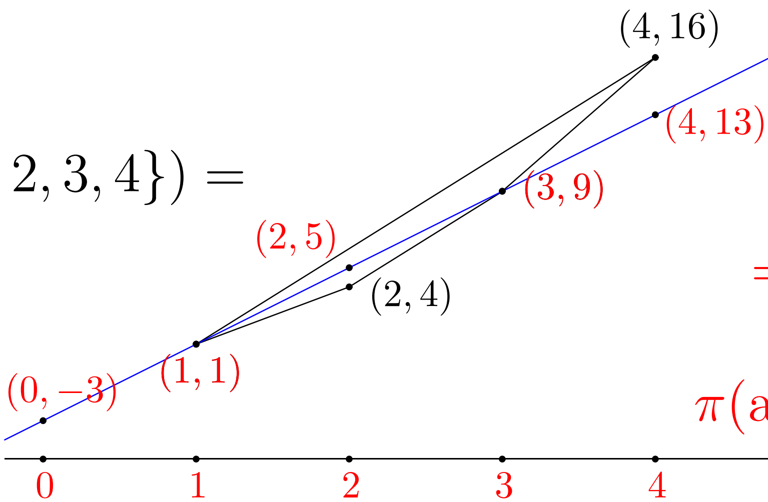

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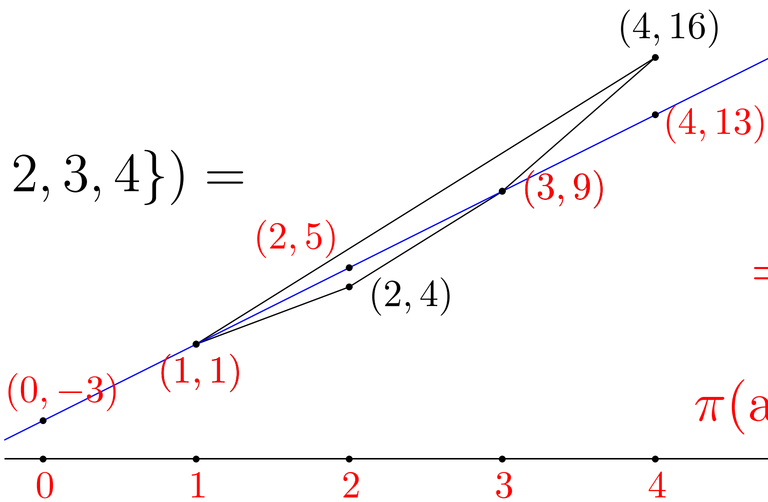

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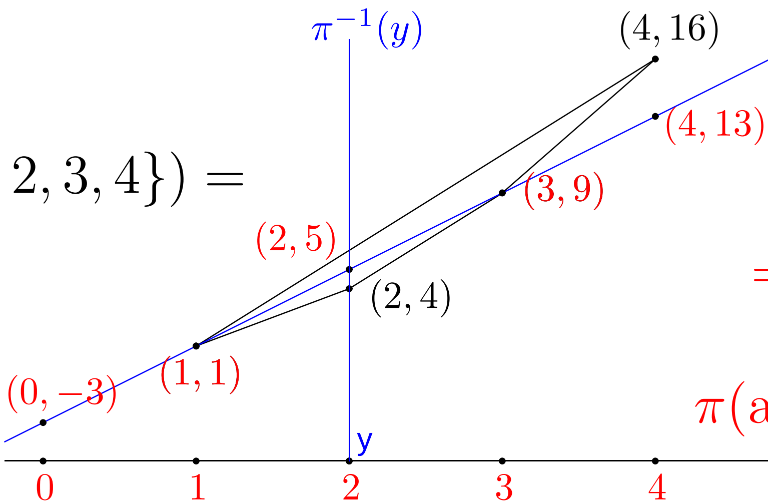
Remark: Condition b) is equivalent to saying that for any lattice point $y \in \mathbb{Z}^{d-1}$, we have that $\pi^{-1}(y) \cap \mathrm{aff}(U)$, the intersection of $\mathrm{aff}(U)$ with the inverse image of y under π, is a lattice point.

Example of condition b): $\pi(\mathrm{aff}(U) \cap \mathbb{Z}^d) = \mathbb{Z}^{d-1}$

Example: $T = \{1, 2, 3, 4\}, d = 2, T' = \{1, 3\}, U = \{(1, 1), (3, 9)\}$.

$P = C_2(\{1, 2, 3, 4\}) =$



$\mathrm{aff}(U) = \{(x, 1 + 4x) \mid x \in \mathbb{R}\}$

$\mathrm{aff}(U) \cap \mathbb{Z}^d$
$= \{\cdots, (0, -3), (1, 1), (2, 5), (3, 9), (4, 13), \cdots\}$

$\pi(\mathrm{aff}(U) \cap \mathbb{Z}^d) = \{\cdots, 0, 1, 2, 3, 4, \cdots, \} = \mathbb{Z}$

Remark: Condition b) is equivalent to saying that for any lattice point $y \in \mathbb{Z}^{d-1}$, we have that $\pi^{-1}(y) \cap \mathrm{aff}(U)$, the intersection of $\mathrm{aff}(U)$ with the inverse image of y under π, is a lattice point.

Definition of lattice-face polytopes

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Lemma 11. *Any lattice-face polytope is an integral polytope.*

Ehrhart polynomials of lattice-face polytopes

Theorem 12 (L)**.** *Let P be a lattice-face d-polytope, then*

$$i(P,m) = \mathrm{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{d-k}(P)) m^k.$$

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Example: Let $d = 3$, let P be the polytope with the vertex set $V = \{v_1 = (0,0,0), v_2 = (4,0,0), v_3 = (3,6,0), v_4 = (2,2,10)\}$. One can check that P is a lattice-face polytope.

$\mathrm{Vol}(P) = 40.$

$\pi(P) = \mathrm{conv}\{(0,0),(4,0),(3,6)\}$, and $\mathrm{Vol}(\pi(P)) = 12.$

$\pi^2(P) = [0,4]$, and $\mathrm{Vol}(\pi^2(P)) = 4.$

Thus, by the theorem, the Ehrhart polynomial of P is

$$i(P,m) = 40m^3 + 12m^2 + 4m + 1.$$

Examples of 2-polytopes

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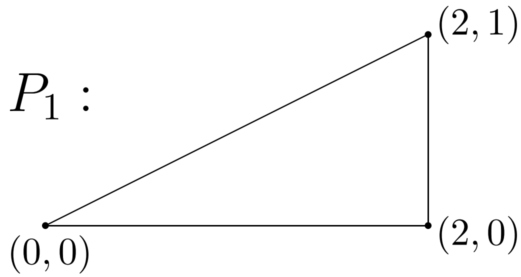

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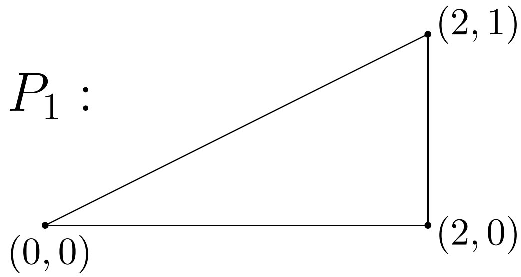

$U_1 = \{v_1, v_2\}, \mathrm{aff}(U_1)$ is $\{(x,0) \mid x \in \mathbb{R}\}$. So $\pi(\mathrm{aff}(U_1) \cap \mathbb{Z}^2) = \mathbb{Z}$.

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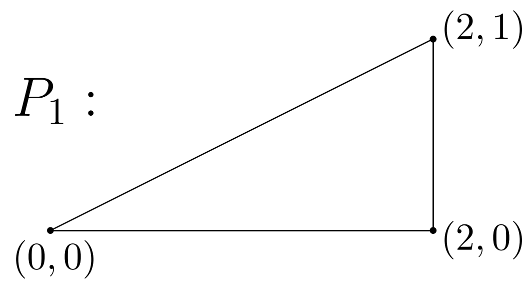

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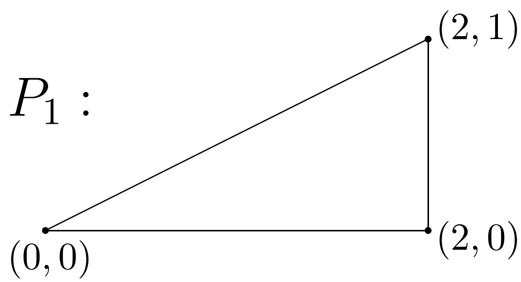

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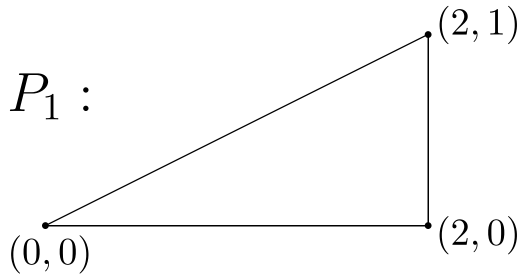

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P_1 is NOT a lattice-face polytope.

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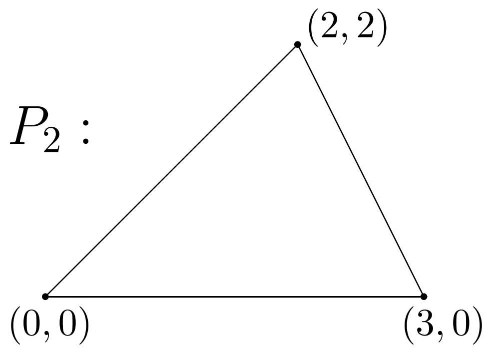
$P_2 :$



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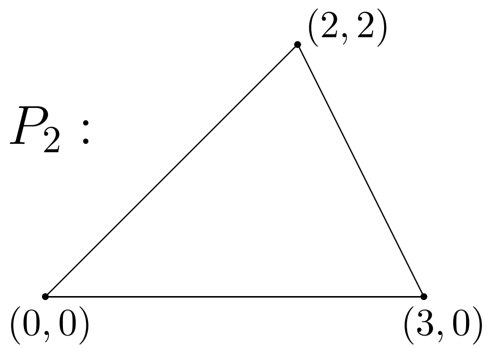

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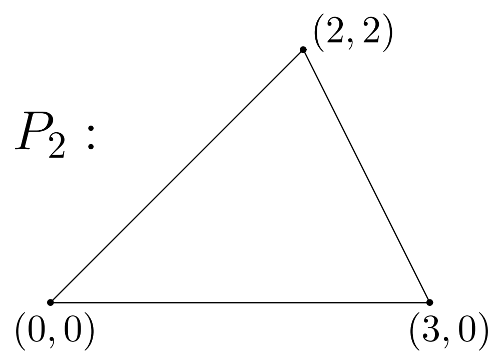

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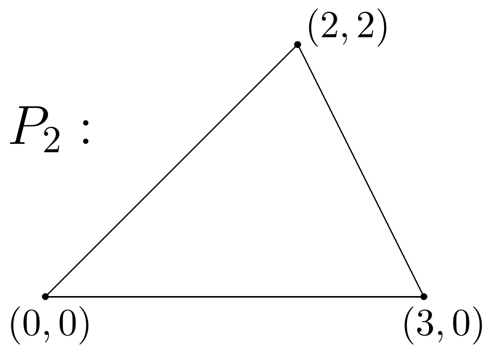

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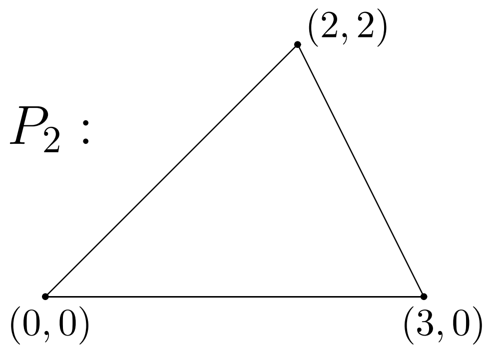

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For each U_i, condition $a)$ is always satisfied.

P_2 is a lattice-face polytope.

How big is the family of lattice-face polytopes?

Theorem 13 (L)**.** *For any rational polytope P, there exists a lattice-face polytope Q having the same combinatorial type as P, that is, the face lattices of P and Q are the same.*

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v. We show that the number of lattice points is given by a formula involving Bernoulli polynomials, signs of permutations, and determinants. By analyzing this formula further, we are able to derive our main theorem.

PART III:

Further discussion

***Summary*:** We give an alternative definition of lattice-face polytopes, which leads us to ask a question and give a conjecture.

An alternative definition

We have an alternative definition of lattice-face polytopes, which is equivalent to the original definition we gave earlier. Indeed, a d-polytope on a vertex set V is a lattice-face polytope if and only if for all $k : 0 \leq k \leq d-1$,

$(\star)$ for any subset $U \subset V$ spanning a k-dimensional space

$$\pi^{d-k}(\mathrm{aff}(U) \cap \mathbb{Z}^d) = \mathbb{Z}^k,$$

In other words, after dropping the last $d-k$ coordinates of the lattice of $\mathrm{aff}(U)$, we get the k-dimensional lattice.

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Note that in this definition, when $k = 0$, satisfying ($\star$) is equivalent to saying that P is an integral polytope, which implies that the last coefficient of the Ehrhart polynomial of P is 1. Therefore, one may ask

Question: If P is a polytope that satisfies ($\star$) for all $k \in K$, where K is a fixed subset of $\{0, 1, \ldots, d-1\}$, can we say something about the Ehrhart polynomial of P?

A conjecture

A special set K can be chosen as the set of consecutive integers from 0 to d', where d' is an integer no greater than $d-1$. Based on some examples in this case, the Ehrhart polynomials seems to follow a certain pattern, so we conjecture the following:

Conjecture 14. *Given $d' \leq d-1$, if P is a d-polytope with vertex set V such that $\forall k : 0 \leq k \leq d'$, (⋆) is satisfied, then for $0 \leq k \leq d'$, the coefficient of m^k in $i(P, m)$ is the same as in $i(\pi^{d-d'}(P), m)$. In other words,*

$$i(P, m) = i(\pi^{d-d'}(P), m) + \sum_{i=d'+1}^{d} c_i m^i.$$

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Example: $P = \mathrm{conv}\{(0,0,0),(4,0,0),(3,6,0),(2,2,2)\}$. One can check that P satisfies ($\star$) for $k = 0, 1$ but not for $k = 2$.

$$i(P,m) = 8m^3 + 10m^2 + 4m + 1,$$

where $4m+1$ is the Ehrhart polynomial of $\pi^2(P) = [0,4]$.