Ehrhart positivity for generalized permutohedra

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Workshop on Computational Commutative Algebra and Convex Polytopes

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This is joint work with Federico Castillo.

Outline

- Introduction
  - Polytopes and Ehrhart positivity
  - Generalized permutohedra and first conjecture
- McMullen's formula and consequences
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  - Reduction theorem and second conjecture
  - Partial results to the conjectures
- The BV-construction and idea of proofs
- Other questions and results

PART I:

Introduction

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Definition. For any polytope $P \subset \mathbb{R}^d$ and positive integer $t \in \mathbb{N}$, the tth dilation of P is $tP = \{t\mathbf{x} : \mathbf{x} \in P\}$. We define

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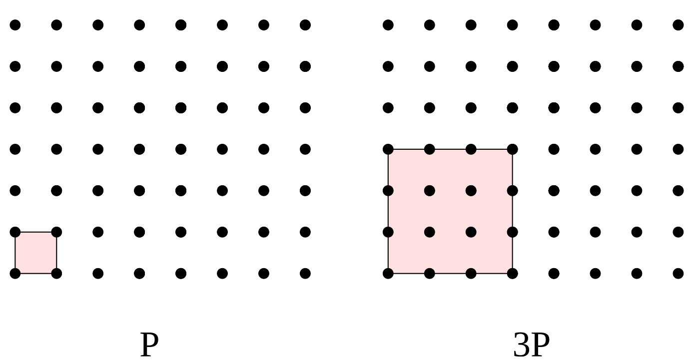
Example: For any d, let $P = \{\mathbf{x} \in \mathbb{R}^d : 0 \le x_i \le 1, \forall i\}$ be the *unit cube* in $\mathbb{R}^d$. Then $tP = \{\mathbf{x} \in \mathbb{R}^d : 0 \le x_i \le t, \forall i\}$ and $i(P,t) = (t+1)^d$.



Theorem of Ehrhart (on integral polytopes)

Theorem 1 (Ehrhart)**.** *Let* P *be a* d*-dimensional integral polytope. Then* $i(P, t)$ *is a polynomial in* t *of degree* d.

Therefore, we call $i(P, t)$ the *Ehrhart polynomial* of P.

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- The constant term of $i(P,t)$ is always 1.
- No simple forms known for other coefficients for general polytopes.
  - It is NOT even true that all the coefficients are positive. For example, for the polytope P with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,13)$, its Ehrhart polynomial is

$$i(P,t) = \frac{13}{6}t^3 + t^2 - \frac{1}{6}t + 1.$$

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We consider *generalized permutohedra*, a family of polytopes that include both Stanley-Pitman polytopes and matroid polytopes.

Conjecture 3 (Castillo-L.)**.** *All integral generalized permutohera are Ehrhart positive.*

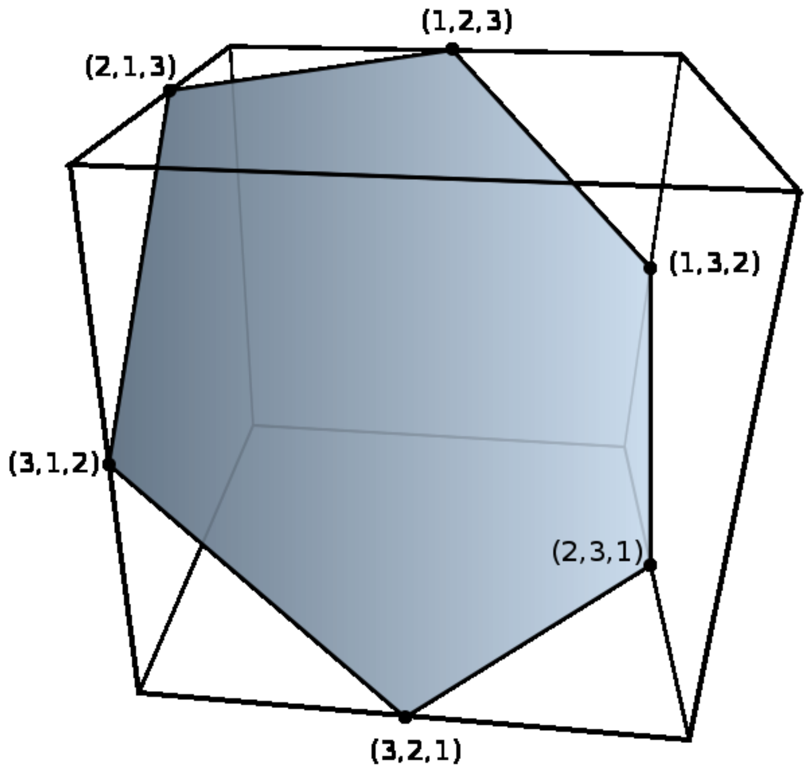
Usual permutohedra

Definition. Suppose $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is a (nondecreasing) sequence. We define the *usual permutohedron*

$$\mathrm{Perm}\,(\mathbf{v}) := \mathrm{conv}\left\{\left(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(n)}\right) \ : \ \sigma \in \mathfrak{S}_n\right\}.$$

- If $\mathbf{v} = (1, 2, \cdots, n)$, we get the *regular permutohedron* Π_{n-1}.

Example. Π_2:



Any usual permutohedron in $\mathbb{R}^n$ is $(n-1)$-dimensional.

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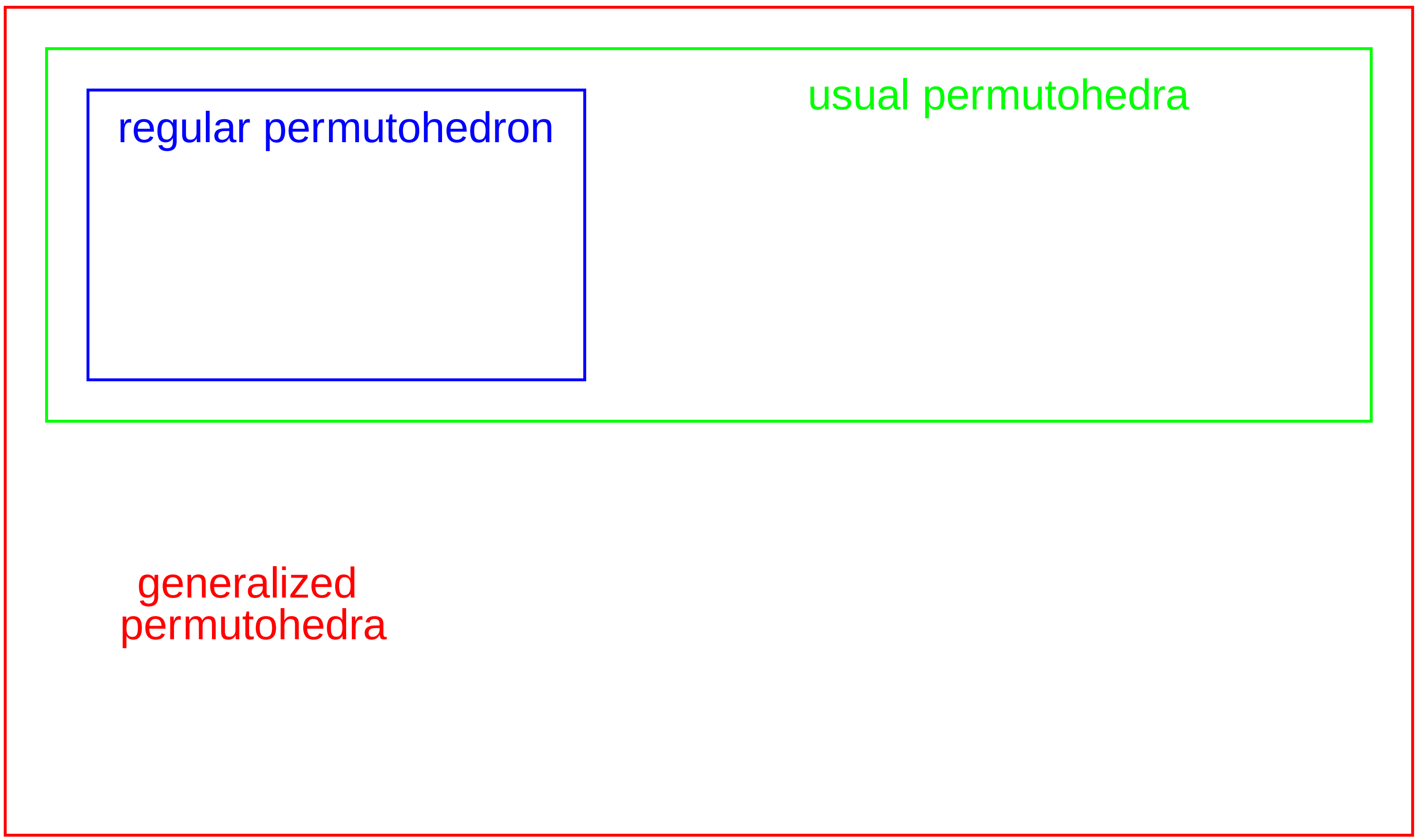

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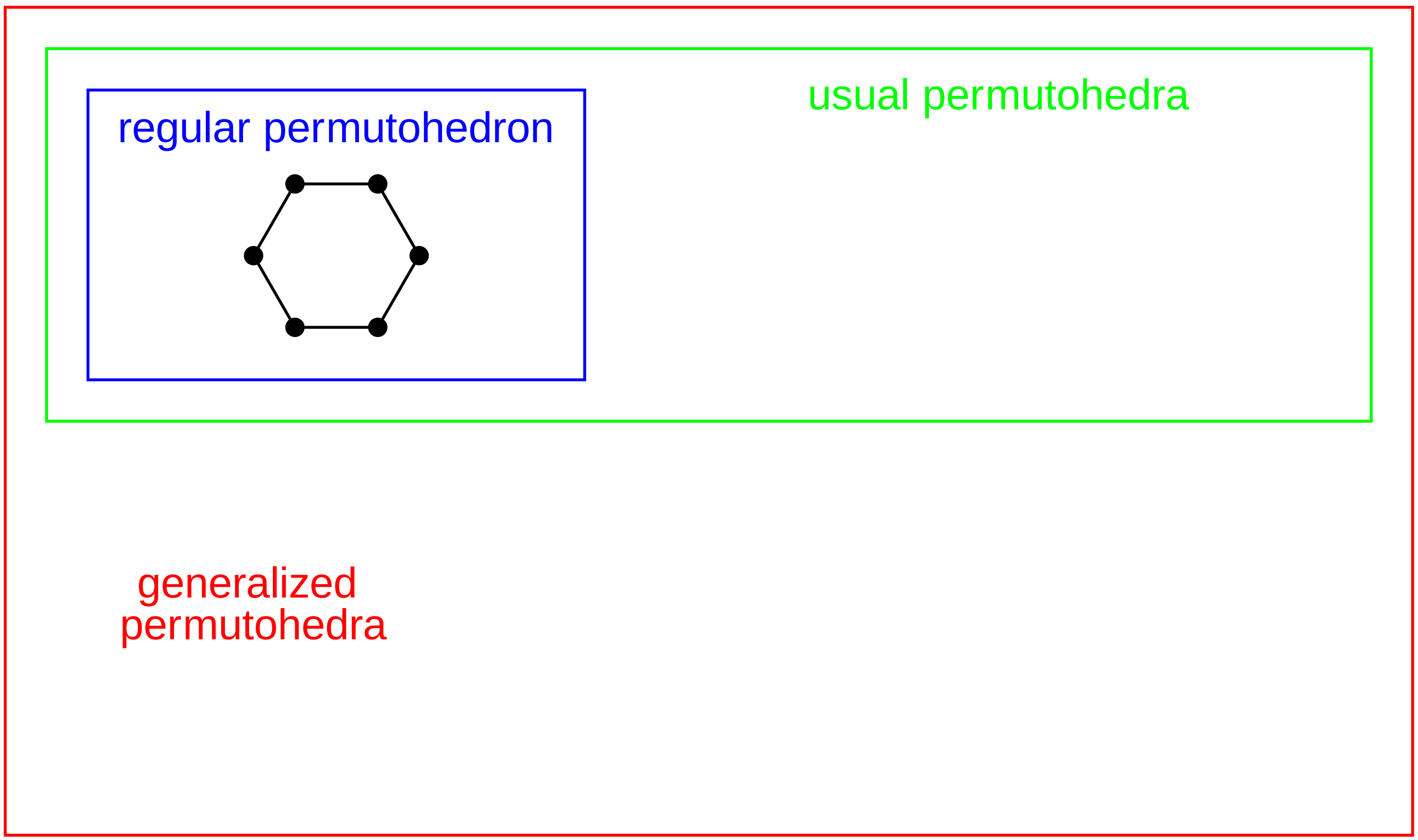

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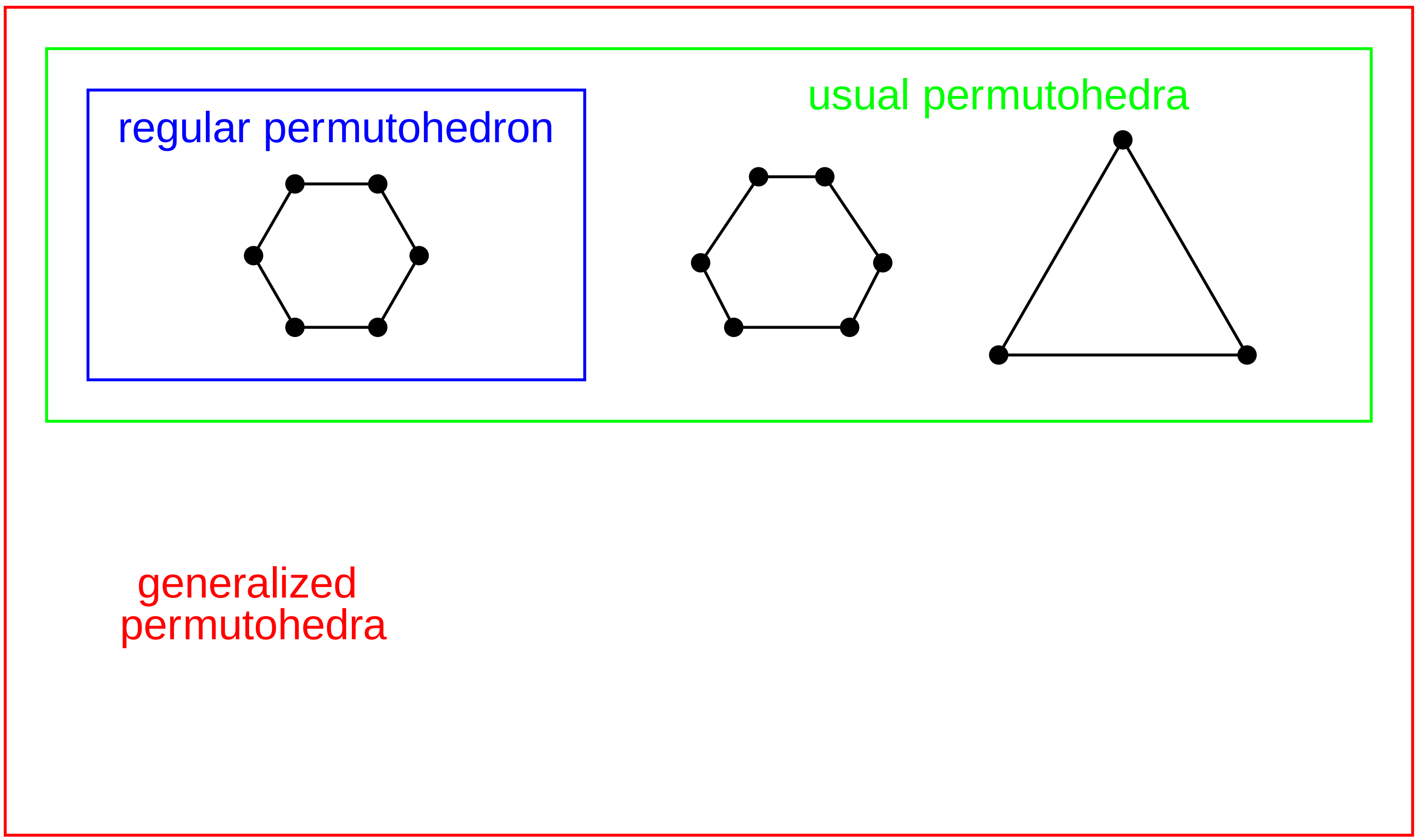

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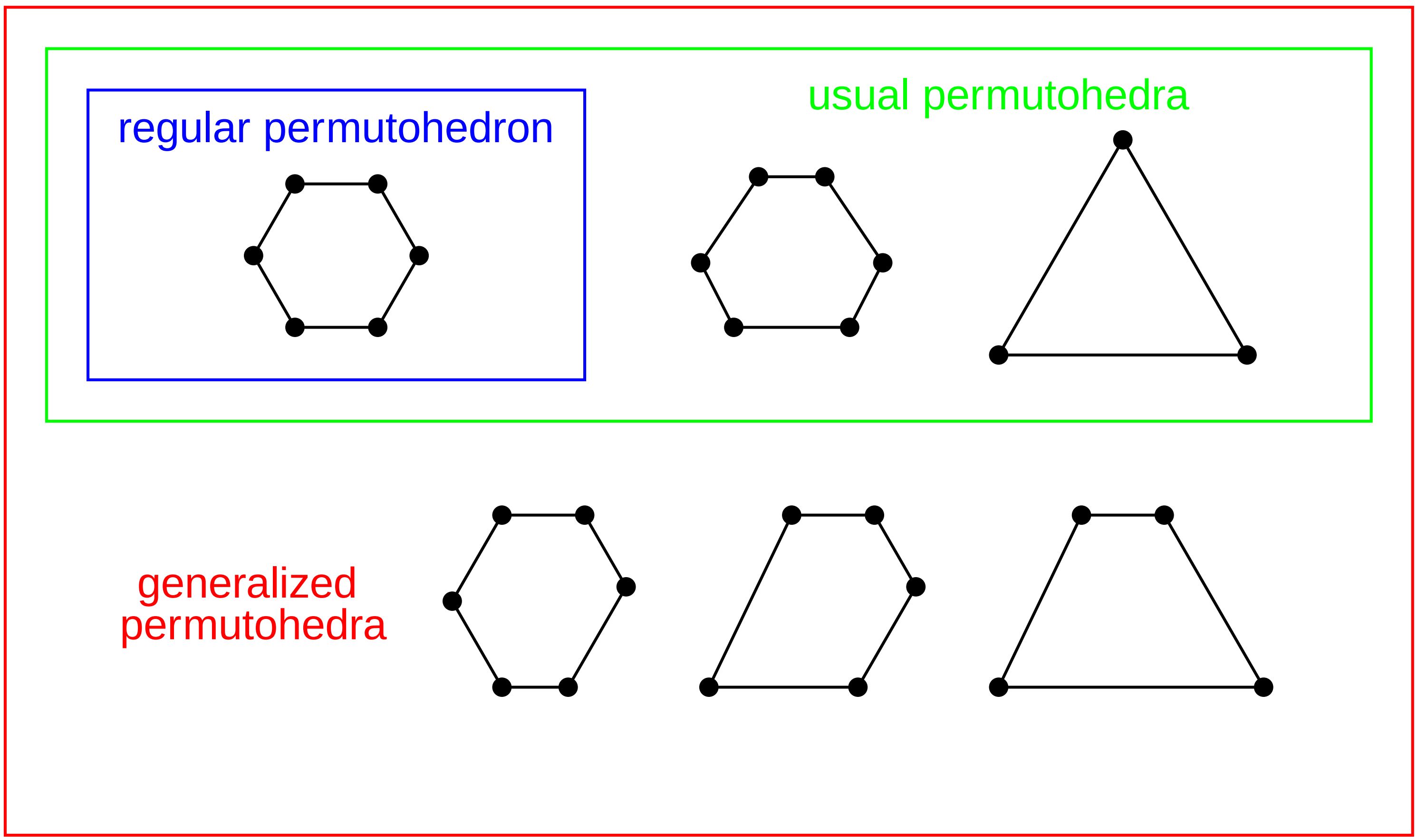

Alternative definition

Let V be the subspace of $\mathbb{R}^n$ defined by $x_1 + x_2 + \cdots + x_n = 0$. The *braid arrangement fan* denoted by B_n, is the complete fan in V given by the hyperplanes

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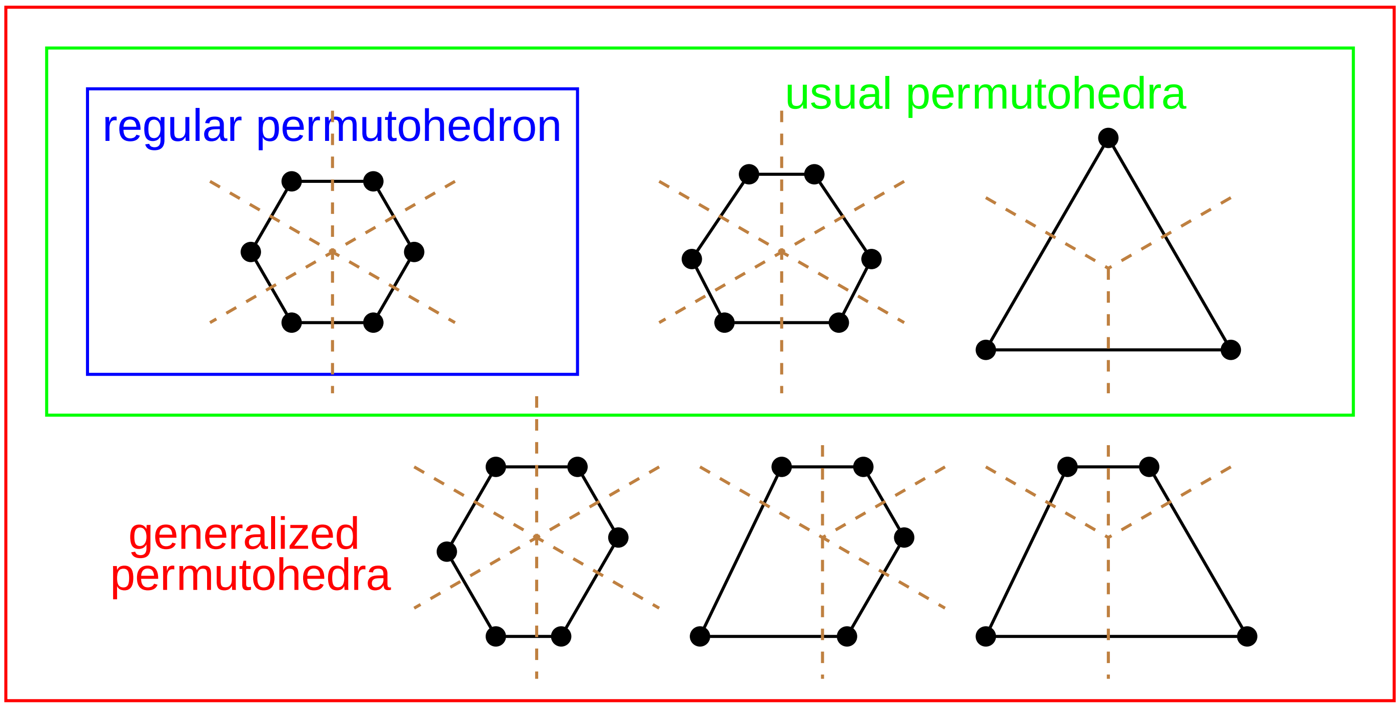

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Postnikov studied the *y-family*, a subset of generalized permutohedra defined by

$$P^{\mathbf{y}} = \sum_{S \subseteq [n]} y_S \Delta_S$$

where

$$\Delta_S = \mathrm{conv}(\mathbf{e}_i : i \in S)$$

and the y_S all nonnegative.

He gave an explicit formula for the Ehrhart polynomial of any polytope in this family.

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Unfortunately, it fails to contain the matroid polytopes.

PART II:

McMullen's formula and consequences

McMullen's formula

Definition. Suppose F is a face of P. The *feasible cone* of P at F, denoted by $\mathrm{fcone}(F, P)$, is the cone of all feasible directions of P at F.

The *pointed feasible cone* of P at F is $\mathrm{fcone}^p(F, P) = \mathrm{fcone}(F, P)/L$, where L is the subspace spanned by F. In general, $\mathrm{fcone}^p(F, P)$ is k-dim pointed cone where if F is codimensional k.

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In 1975 Danilov asked if it is possible to assign values $\Psi(C)$ to all rational cones C such that the following *McMullen's formula* holds

$$|P \cap \mathbb{Z}^d| = \sum_{F:\text{ a face of } P} \alpha(F, P)\,\mathrm{vol}(F).$$

where $\alpha(F, P) := \Psi(\mathrm{fcone}^p(F, P))$.

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- McMullen proved it was possible in a non constructive way.
- Subsequently, explicit constructions of $\Psi/\alpha(F, P)$ were given by Morelli, Pommersheim-Thomas, and Berline-Vergne.

We will use Berline-Vergne's construction, which we will refer to as the *BV-construction*.

An expression for Ehrhart coefficients

Given an integral polytope $P \subseteq \mathbb{R}^d$, any dilation tP of P is integral as well. We have

$$\begin{aligned} i(P,t) = |tP \cap \mathbb{Z}^d| &= \sum_{F\text{: a face of } P} \alpha(tF, tP)\,\mathrm{vol}(tF) \\ &= \sum_{F\text{: a face of } P} \alpha(F, P)\,\mathrm{vol}(F) t^{\dim(F)} \end{aligned}$$

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Hence, the coefficient of t^k is positive if $\alpha(F,P)$ is positive for any k-dimensional face F of P.

Moreover, as long as all α for P are positive, P is Ehrhart positive.

Reduction Theorem

For the rest of this part, we assume that α is the BV-construction.

Theorem 5 (Castillo-L.)**.** *Suppose* $\alpha(F, \Pi_{n-1}) > 0$ *for any* k*-dimensional face* F *of the regular permutohedron* Π_{n-1}*. Then* $\alpha(G, Q) > 0$ *for any* k*-dimensional face* G *of any generalized permutohedron* Q *in* $\mathbb{R}^n$*.*

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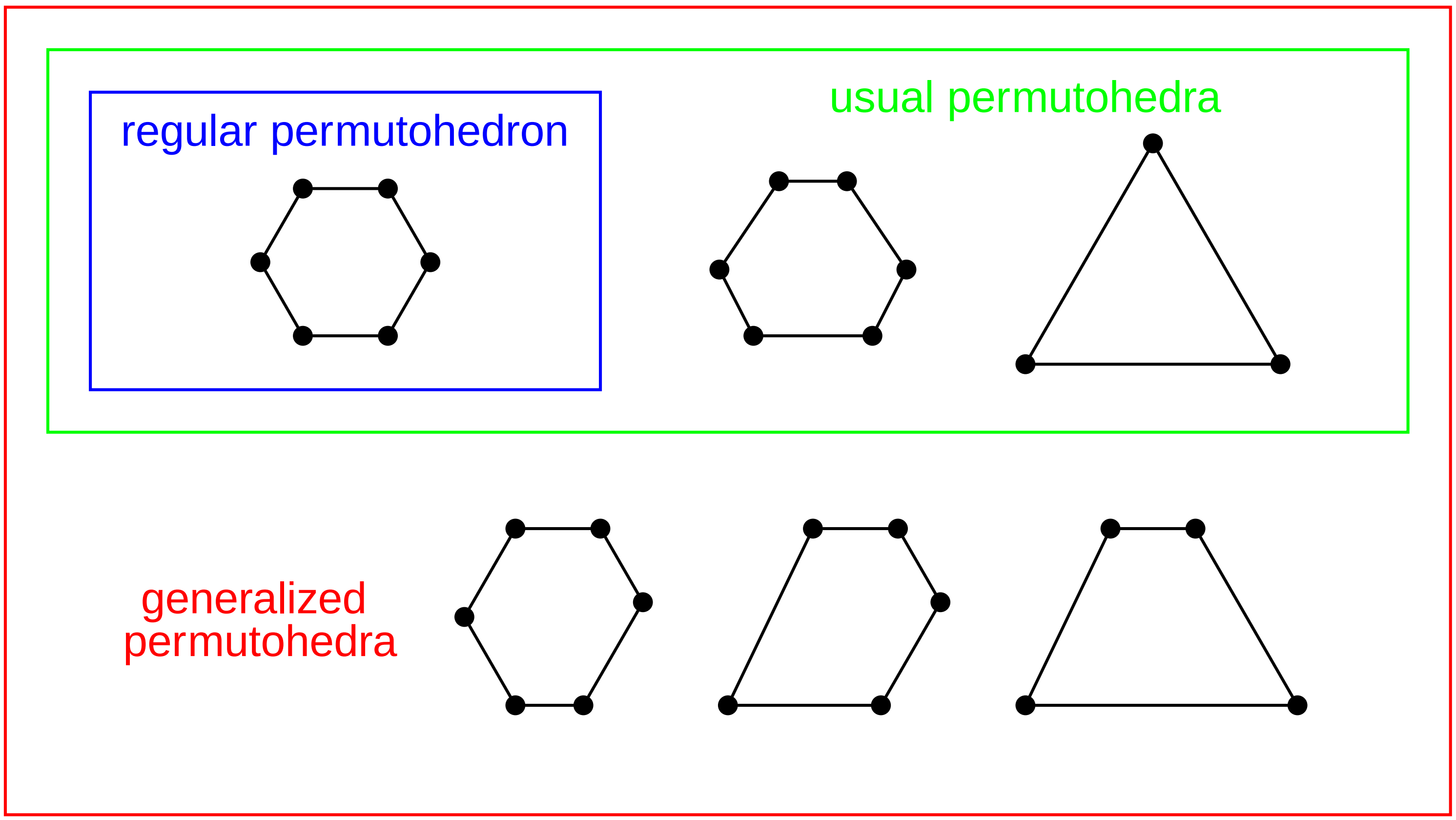

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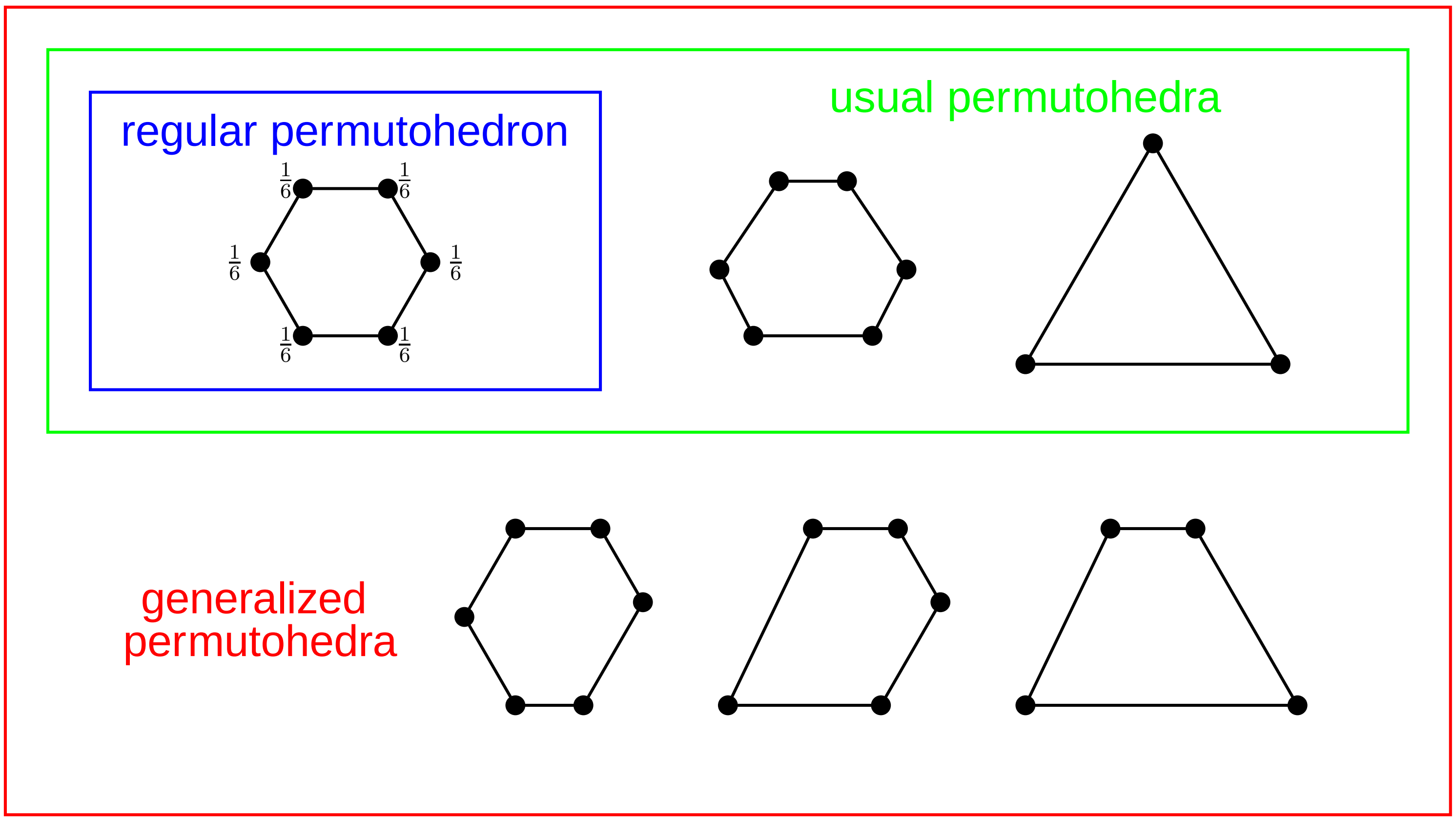

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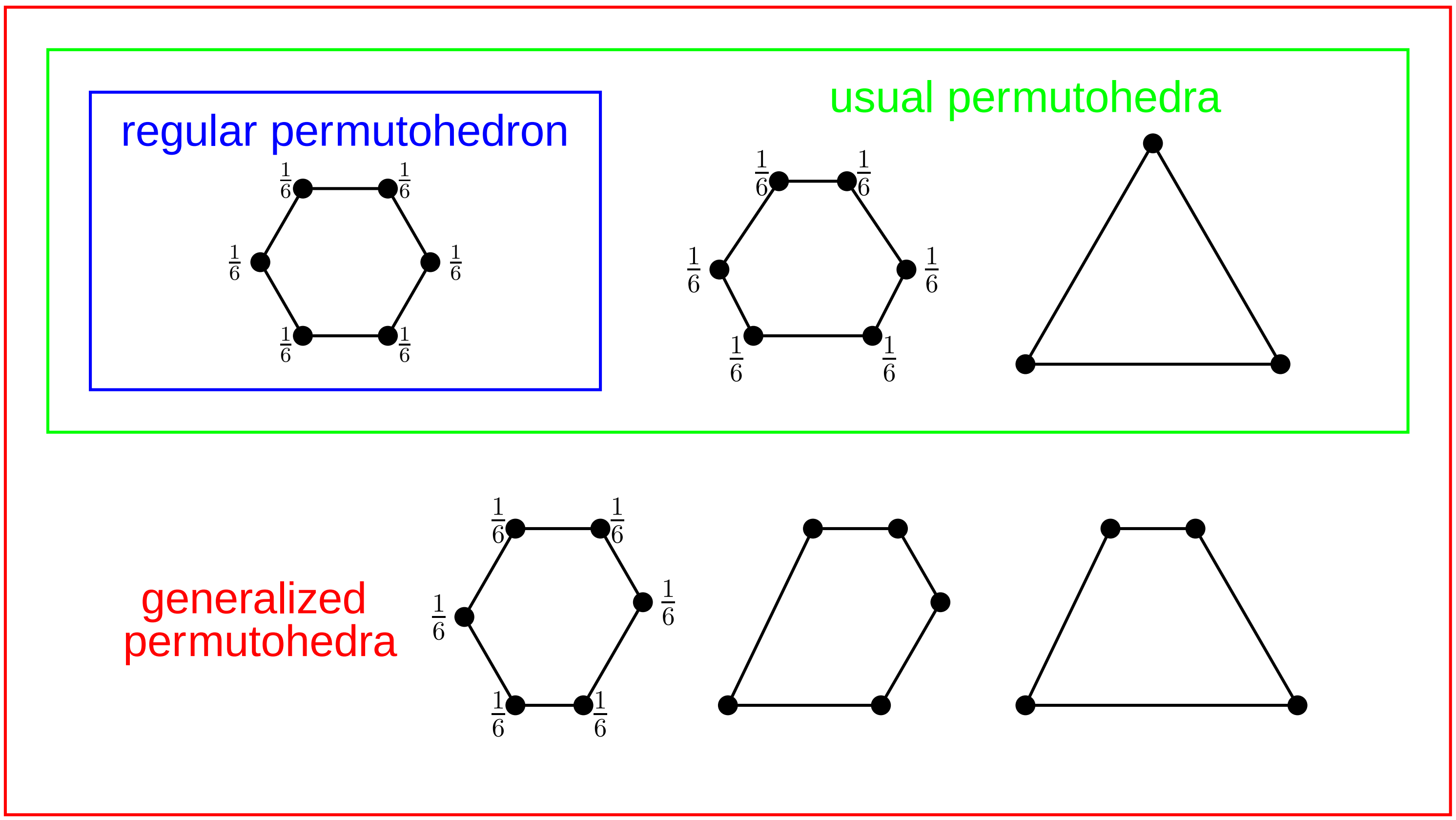

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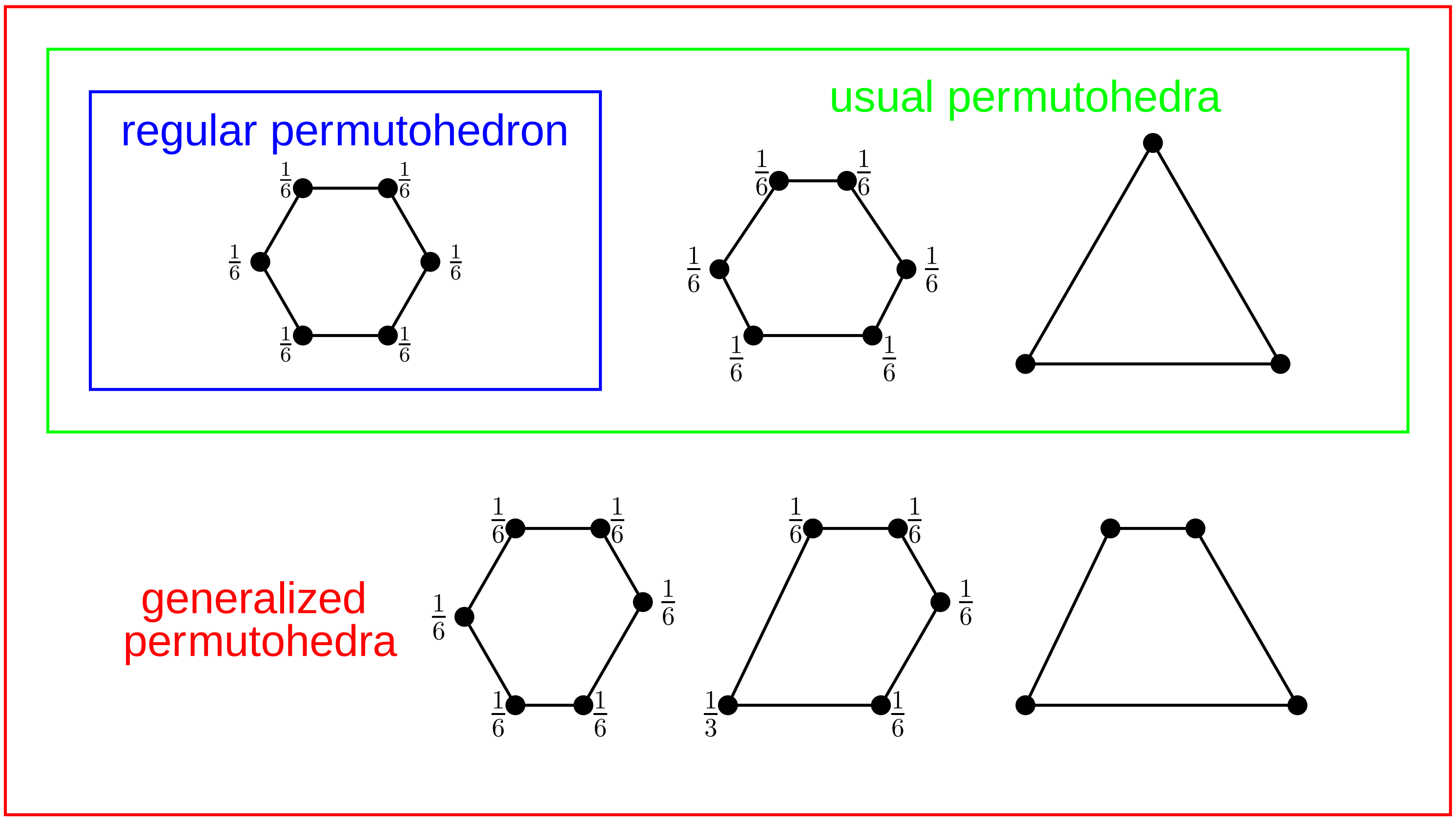

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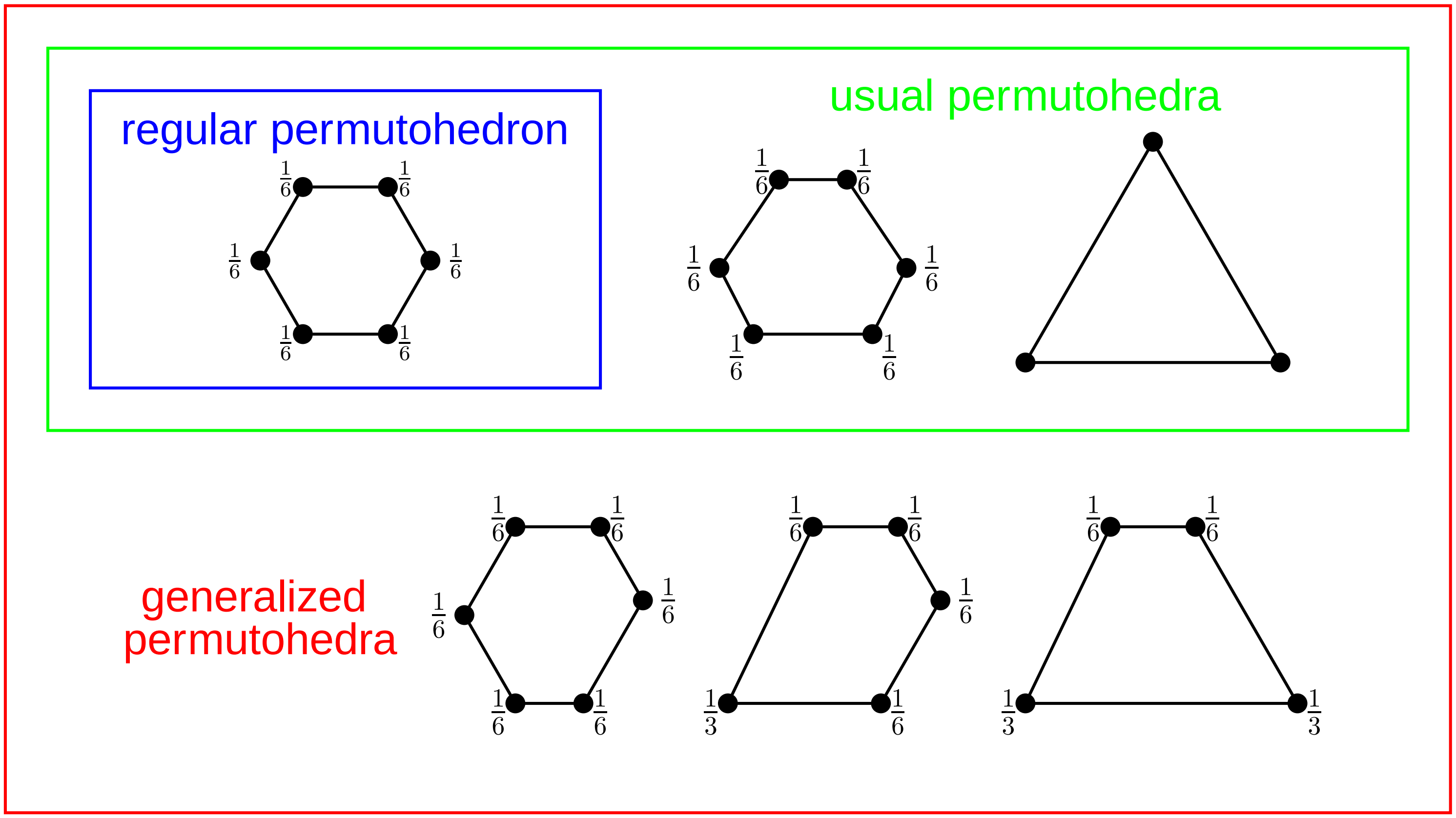

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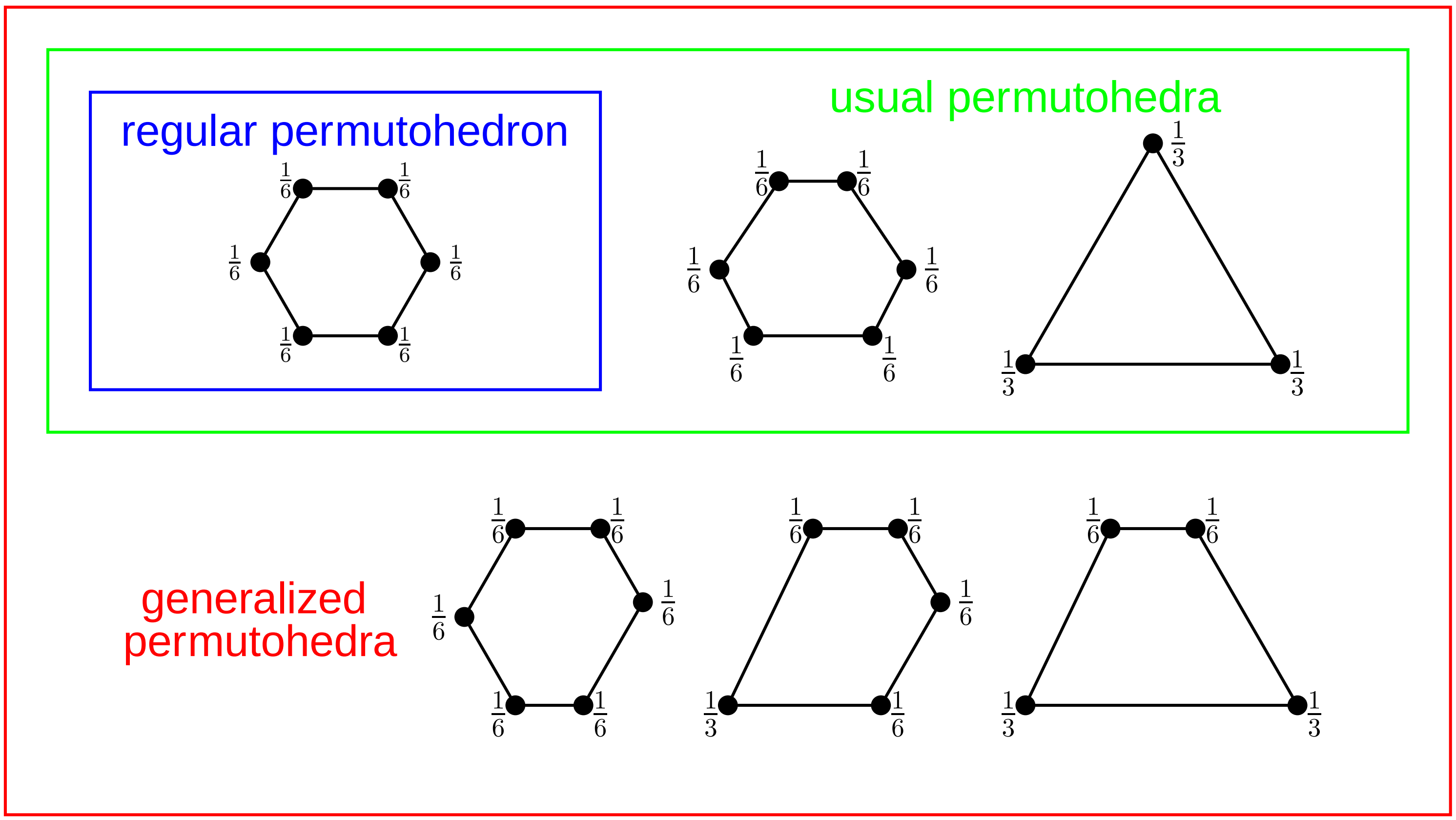

A stronger conjecture

Conjecture 6 (Castillo-L.). *The α values (from the BV-construction) of the regular permutoheron Π_{n-1} are all positive.*

This conjecture clearly implies our first conjecture by the reduction theorem.

Note. The “regular permutohedron Π_{n-1}” can be replaced with “any generalized permutohedron whose normal fan is the braid arrangement fan B_n”.

Thus we may state this conjecture as “the α values for B_n are all positive”.

A more general form of the reduction theorem

The reduction theorem does not only work for Π_{n-1} and generalized permutohedra.

Theorem 7 (Castillo-L.)**.** *Suppose Q is a deformation of P, or the normal fan of P is a refinement of the normal fan of Q. If $\alpha(F, P) > 0$ for any k-dimensional face F of P, then $\alpha(G, Q) > 0$ for any k-dimensional face G of Q.*

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Applying the reduction theorem, we get:

Corollary (Castillo-L.). *i. Any integral generalized permutohedron of dimension ≤ 6 is Ehrhart positive.*

ii. The third and fourth coefficients in the Ehrhart polynomial of any integral generalized permutohedron is positive.

iii. The linear coefficient in the Ehrhart polynomial of any integral generalized permutohedron of dimension ≤ 100 is positive.

PART III:

The BV-construction and idea of proofs

Computing the BV-construction

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Lemma 8. *Let C be a* ***one-dimensional*** *(unimodular) cone. Then $\Psi(C) = 1/2$.*

Computing the BV-Construction (cont'd)

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Lemma 9. *If* $C = \mathrm{Cone}(\mathbf{u}_1, \mathbf{u}_2)$, *where* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *is a basis for the lattice* $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2) \cap \mathbb{Z}^n$, *then*

$$\Psi(C) = \frac{1}{4} + \frac{1}{12}\left(\frac{\langle \mathbf{u}_1, \mathbf{u}_2\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle} + \frac{\langle \mathbf{u}_1, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\right).$$

Computing the BV-Construction (cont'd)

Lemma 9. *If* $C = \mathrm{Cone}(\mathbf{u}_1, \mathbf{u}_2)$, *where* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *is a basis for the lattice* $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2) \cap \mathbb{Z}^n$, *then*

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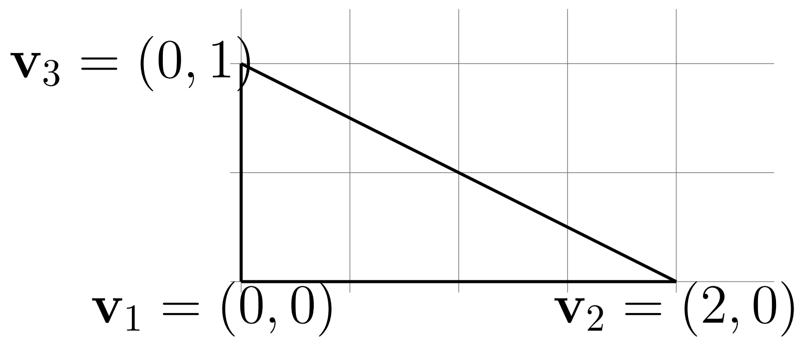
Example. Consider the polygon P in $\mathbb{R}^2$ with vertices $\mathbf{v}_1 = (0,0)$, $\mathbf{v}_2 = (2,0)$, and $\mathbf{v}_3 = (0,1)$. Let $C_i = \mathrm{fcone}^p(\mathbf{v}_i, P)$.



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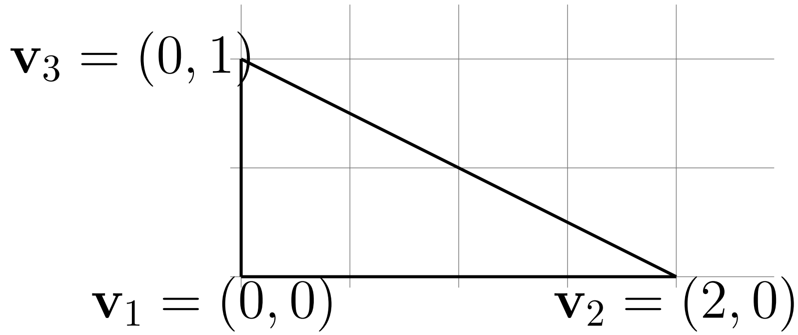

$C_1 = \mathrm{Cone}((1,0),(0,1))$ is a unimodular cone. Thus,

$$\alpha(\mathbf{v}_1, P) = \Psi(C_1) = \frac{1}{4} + \frac{1}{12}\left(\frac{0}{1} + \frac{0}{1}\right) = \frac{1}{4}.$$

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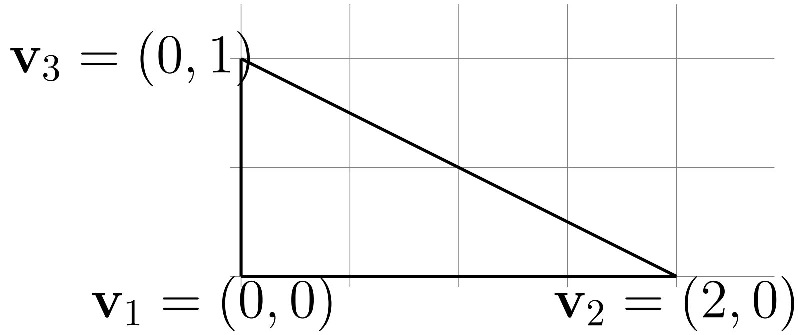

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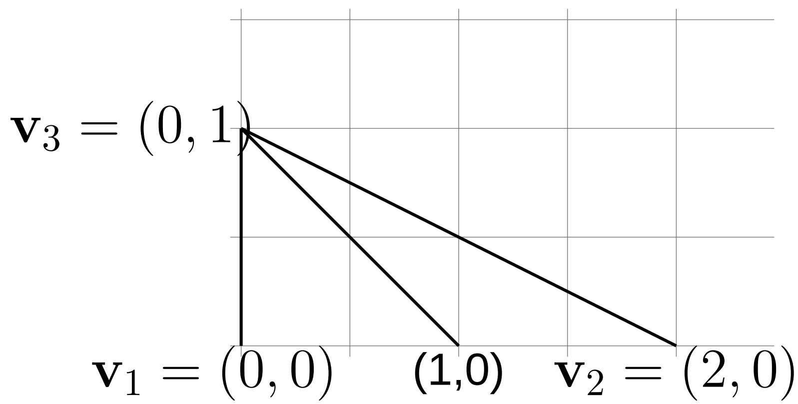

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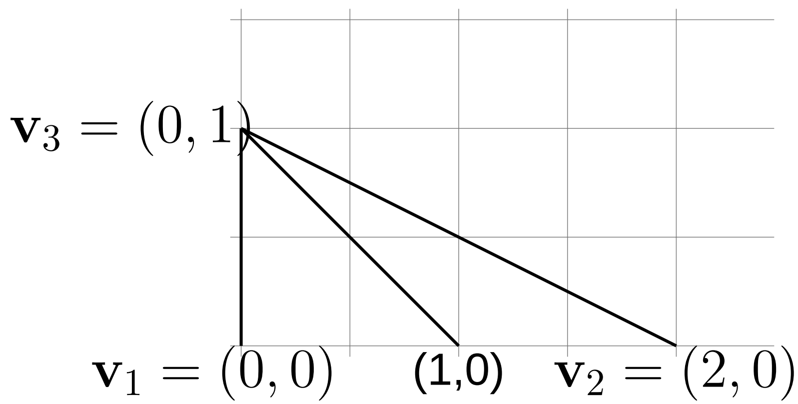

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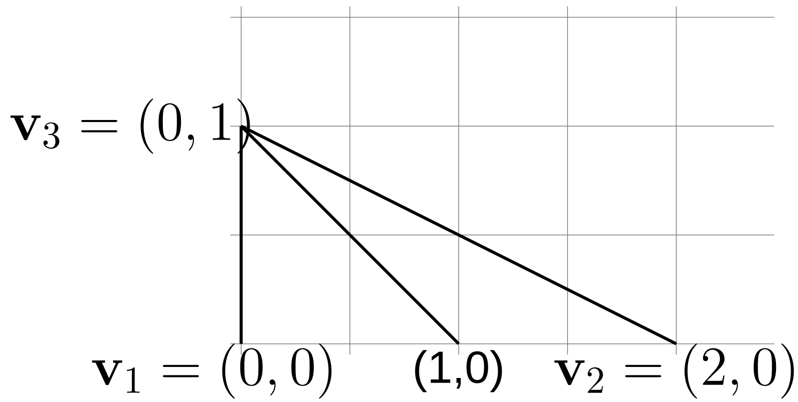

We apply the formula to the two first cones in the above decomposition and get Ψ values of $3/8$ and $17/40$. Since the last cone is one-dimensional, we get its Ψ value to be $1/2$. Finally, by Ψ is a valuation function, we get

$$\alpha(\mathbf{v}_3, P) = \Psi(C_3) = \frac{3}{8} + \frac{17}{40} - \frac{1}{2} = \frac{3}{10}.$$

Computing the BV-Construction (cont'd)

Lemma 10. *If* $C = \mathrm{Cone}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ *where* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *is a basis for the lattice* $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2) \cap \mathbb{Z}^n$, *then*

$$\Psi(C) = \frac{1}{8} + \frac{1}{24}\left(\frac{\langle \mathbf{u}_1, \mathbf{u}_2\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle} + \frac{\langle \mathbf{u}_1, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle} + \frac{\langle \mathbf{u}_1, \mathbf{u}_3\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle} + \frac{\langle \mathbf{u}_1, \mathbf{u}_3\rangle}{\langle \mathbf{u}_3, \mathbf{u}_3\rangle} + \frac{\langle \mathbf{u}_3, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle} + \frac{\langle \mathbf{u}_3, \mathbf{u}_2\rangle}{\langle \mathbf{u}_3, \mathbf{u}_3\rangle}\right).$$

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Fact 12. Ψ is computed recursively. So lower dimensional cones are easier to compute.

Recall that if F is a codimension k face of P, then $\mathrm{fcone}^p(F, P)$ is k-dimensional. Thus, $\alpha(F, P)$ is easier to compute if F is a higher dimensional face.

Proofs of lemmas

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Remark. The number "100" in the lemma can be pushed further.

The symmetry property

Lemma. *The valuation* Ψ *(from the BV-construction) is* symmetric about the coordinates*, i.e., for any cone* $C \in \mathbb{R}^n$ *and any signed permutation* $(\sigma, \mathbf{s}) \in \mathfrak{S}_n \times \{\pm 1\}^n$*, we have*

$$\Psi(C) = \Psi((\sigma, \mathbf{s})(C)),$$

where $(\sigma, \mathbf{s})(C) = \{(s_1 x_{\sigma(1)}, s_2 x_{\sigma(2)}, \dots, s_n x_{\sigma(n)}) : (x_1, \dots, x_n) \in C\}.$

Idea of the proof of Lemma 13

Recall that the coefficient of t^k in $i(P,t)$ is given by

$$\sum_{F:\text{ a } k\text{-dimensional face of } P} \alpha(F,P)\,\mathrm{vol}(F).$$

In particular, the coefficient of the linear term is given by

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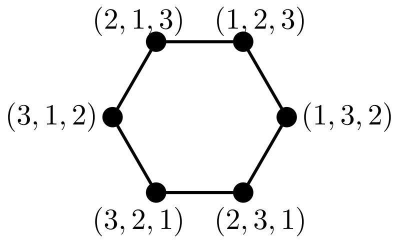
General idea: Suppose you have a family of polytopes such that

- they have same pointed feasible cones (for edges) up to signed permutations, and thus have the same α-values;
- the Ehrhart polynomial of each polytope in the family is known (or at least the linear Ehrhart coefficient is known).

Then as long as you have enough “independent” polytopes in your family, you can figure out the α-values.

Idea of the proof of Lemma 13 (cont'd)

Example. When $n = 3$: $\Pi_2 = \mathrm{Perm}((1,2,3)) = \mathrm{conv}\{\sigma \ : \ \sigma \in \mathfrak{S}_3\}$.



The pointed feasible cones of the six edges of Π_2 are

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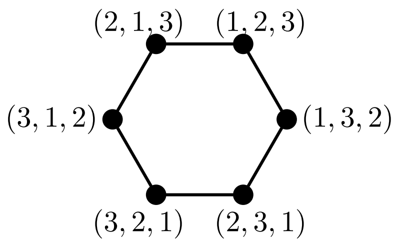

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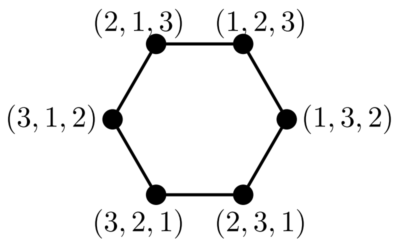

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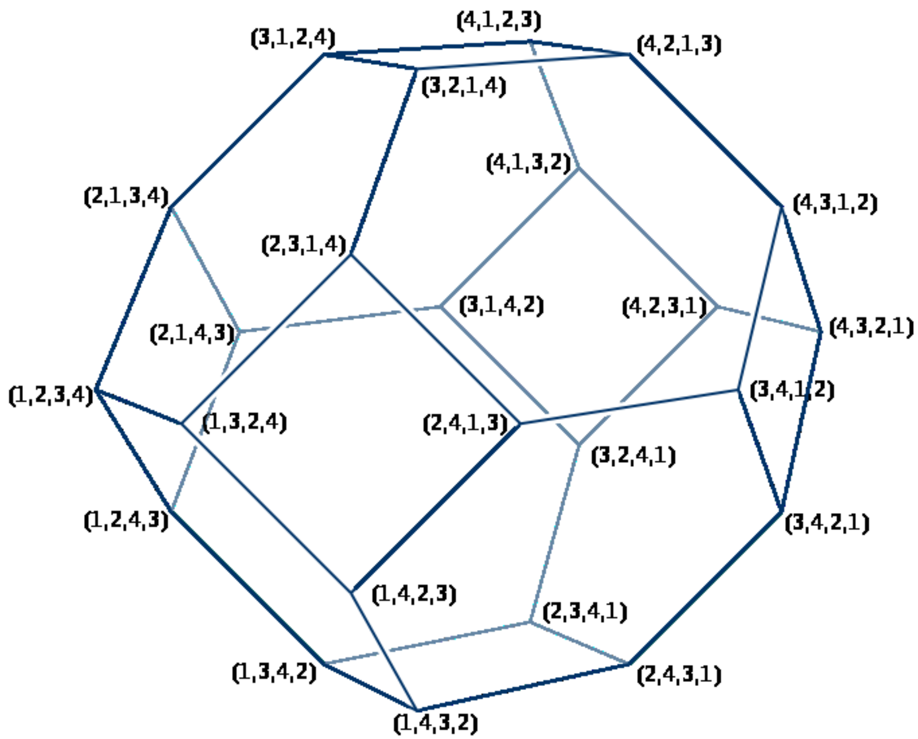
The Ehrhart polynomial of Π_2 is $3t^2 + 3t + 1$. Thus,

$$3 = \sum_E \alpha(E, \Pi_2) \cdot \mathrm{vol}(E) = 6\alpha \quad \Rightarrow \quad \alpha = 1/2 > 0.$$

Idea of the proof of Lemma 13 (cont'd)

Example. When $n = 4$: $\Pi_3 = \mathrm{Perm}((1,2,3,4)) = \{\sigma \; : \; \sigma \in \mathfrak{S}_4\}$.



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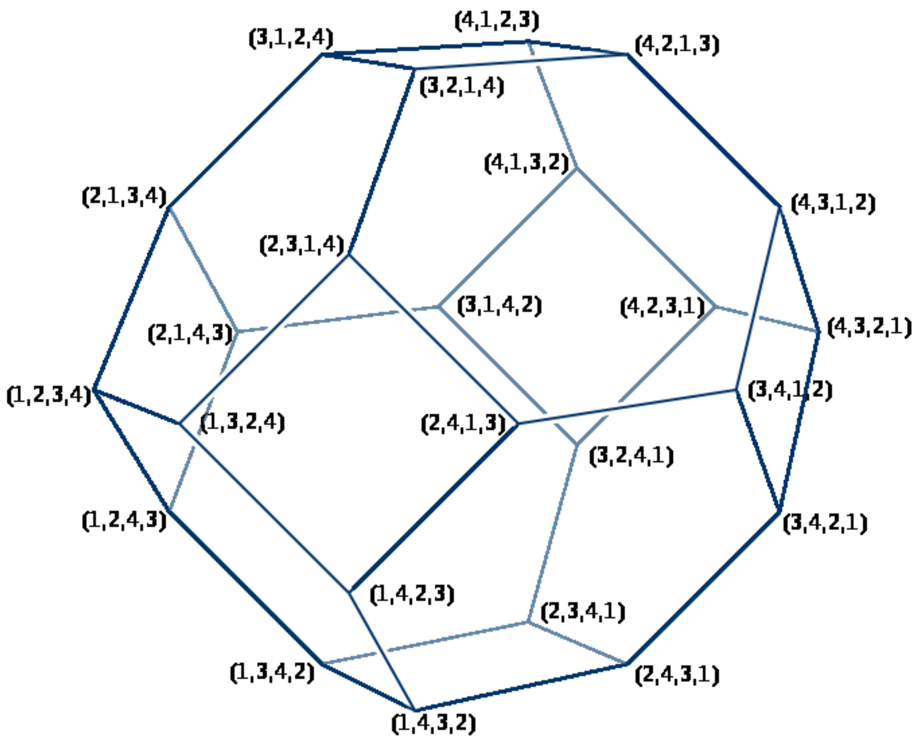

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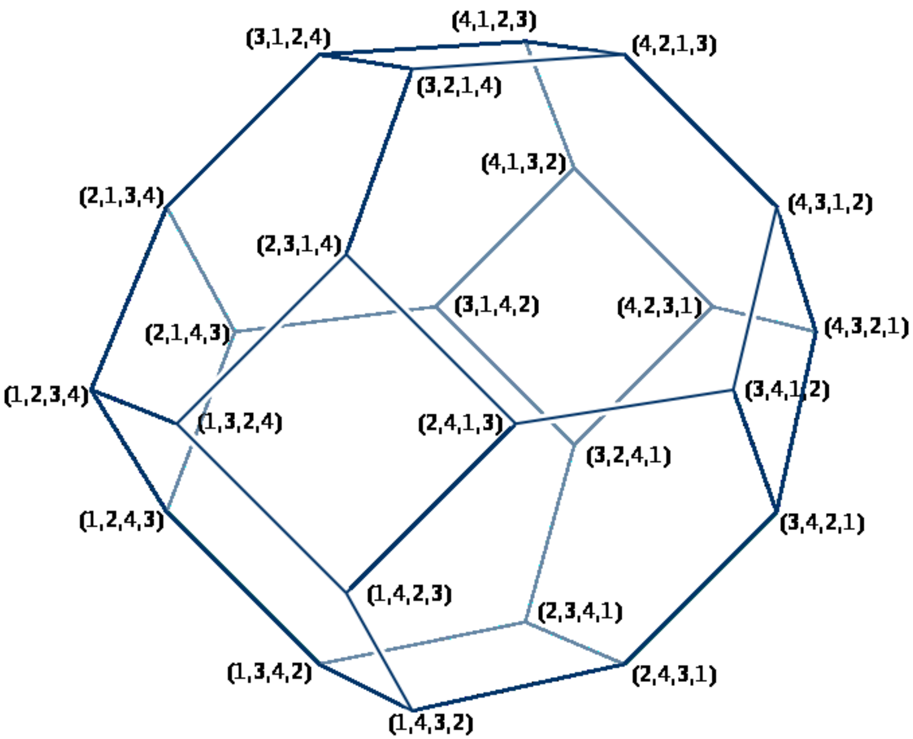

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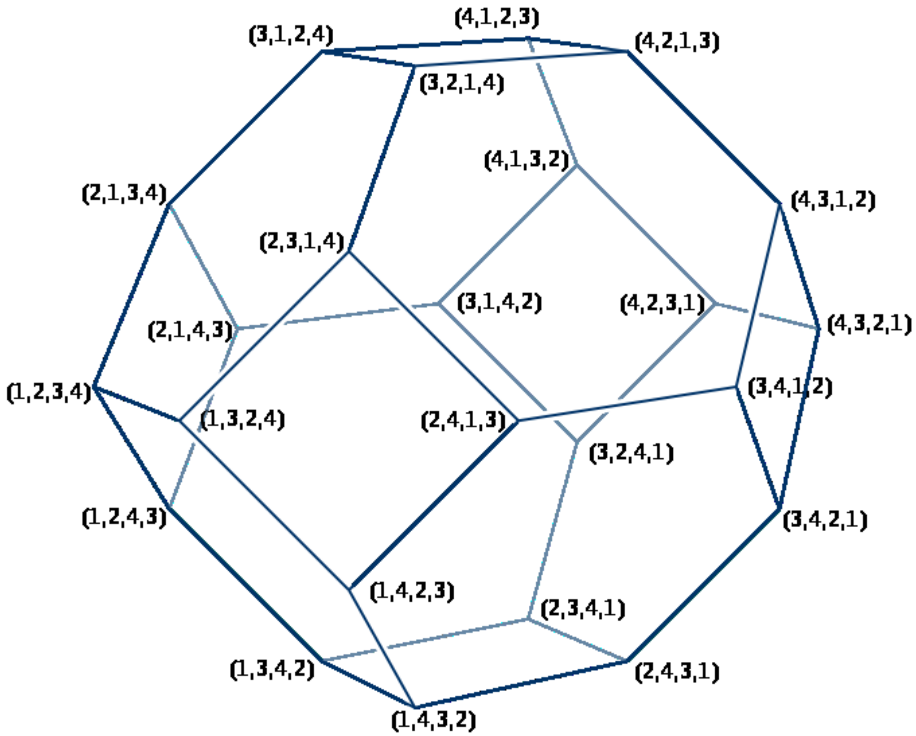

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Not enough equations!

Idea of the proof of Lemma 13 (cont'd)

Consider the hypersimplex $\Delta_{2,4} = \mathrm{Perm}((0,0,1,1))$. It has 12 edges whose corresponding pointed feasible cones are the same as that of the 12 long edges of Π_3. So they all have α-values α_2.

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The Ehrhart polynomial of $\Delta_{2,4}$ is $\frac{2}{3}t^3 + 2t^2 + \frac{7}{3}t + 1$. Thus,

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For arbitrary n: The linear Ehrhart coeffcient of some polytopes in the y-family can be easily described. Using these, we were able to set up an explicit triangular linear system for $\{\alpha(E, \Pi_{n-1}) \ : \ E \text{ is an edge of } \Pi_{n-1}\}$ for any n.

PART IV:

Other questions and results

Uniqueness of the construction of Ψ

The solution Ψ to McMullen's formula is **not** unique since we know there are different constructions.

Observation 14. When we prove Lemma 13, we did not really compute Berline-Vergne's construction. Instead, we just use the fact that their construction is symmetric about the coordinates to set up linear system to solve.

E.g., in the case of Π_3 we did in the last example, as long as we know a construction Ψ

- satisfies McMullen's formula, and
- is symmetric about the coordinates,

we will set up exactly the same 2×2 linear system, and find exactly the same two α-values.

So Ψ of the cones appeared in the example or the values of $\alpha(E, \Pi_3)$ are unique.

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Idea of proof: Use mixed Ehrhart theory.

Mixed Ehrhart Theorem

Consider the following Minkwoski sum:

$$P = w_1 P_1 + w_2 P_2 + \cdots + w_k P_k,$$

where w_i are variables and P_i are polytopes.

Mixed Ehrhart Theorem The number of integer points in P is a polynomial in w_i's.

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Mixed Ehrhart Theorem The number of integer points in P is a polynomial in w_i's. The coefficients are called *mixed Ehrhart coefficients*.

Postnikov showed that usual permutohedra are Minkowski sums of hypersimplices.

$$\mathrm{Perm}(\mathbf{v}) = w_1 \Delta_{1,n} + w_2 \Delta_{2,n} + \cdots + w_{n-1} \Delta_{n-1,n},$$

where

$$w_i := v_{i+1} - v_i \text{ for } i = 1, 2, \ldots, n-1,$$

and the *hypersimplex* $\Delta_{k,n}$ is defined as

$$\Delta_{k,n} = \mathrm{Perm}(\underbrace{0, \cdots, 0}_{n-k}, \underbrace{1, \cdots, 1}_{k}).$$

A formula

Theorem 17 (Castillo-L.)**.** *Suppose* Ψ *is a solution to McMullen's formula and is symmetric about the coordinates. Then the* α *values for the regular permutohedron* Π_{n-1} *(or the braid arrangement fan* B_n*) are positive scalars of mixed Ehrhart coefficients of hypersimplices.*

Consequences

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ii. The following three statements are equivalent:

(a) All α values of Π_{n-1} are positive. (The strong conjecture).

(b) All mixed Ehrhart coefficients of hypersimplices are positive.

(c) Let X be corresponding toric variety to the braid arrangement fan B_n. The Todd class is positive with respect to the torus invariant cycles, that is

$$\mathrm{Todd}(X) = \sum_{\sigma \in B_n} r_\sigma [V(\sigma)],$$

for some $r_\sigma > 0$.