Ehrhart positivity for generalized permutohedra

Fu Liu

University of California, Davis

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This is joint work with Federico Castillo.
Outline

• Introduction
  – Polytopes and Ehrhart positivity
  – Generalized permutohedra and first conjecture

• McMullen’s formula and consequences
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  – Reduction theorem and second conjecture
  – Partial results to the conjectures

• The BV-construction and idea of proofs

• Other questions and results
PART I:

Introduction
A *(convex) polytope* is a bounded solution set of a finite system of linear inequalities, or is the convex hull of a finite set of points.
Lattice points of a polytope

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\textbf{Definition}. For any polytope $P \subset \mathbb{R}^d$ and positive integer $t \in \mathbb{N}$, the \textit{$t$th dilation of $P$} is $tP = \{tx : x \in P\}$. We define

$$i(P, t) = |tP \cap \mathbb{Z}^d|$$

to be the number of lattice points in the $tP$. 


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to be the number of lattice points in the $tP$.

**Example:** For any $d$, let $P = \{x \in \mathbb{R}^d : 0 \leq x_i \leq 1, \forall i\}$ be the unit cube in $\mathbb{R}^d$. Then $tP = \{x \in \mathbb{R}^d : 0 \leq x_i \leq t, \forall i\}$ and $i(P, t) = (t + 1)^d$. 
Theorem of Ehrhart (on integral polytopes)

**Theorem 1** (Ehrhart). Let $P$ be a $d$-dimensional integral polytope. Then $i(P,t)$ is a polynomial in $t$ of degree $d$.

Therefore, we call $i(P,t)$ the *Ehrhart polynomial* of $P$. 
Coefficients of Ehrhart polynomials

If $P$ is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, t)$?
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- The second coefficient equals $1/2$ of the sum of the normalized volumes of each facet.
- The constant term of $i(P, t)$ is always $1$.
- No simple forms known for other coefficients for general polytopes.
  - It is NOT even true that all the coefficients are positive. For example, for the polytope $P$ with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(1, 1, 13)$, its Ehrhart polynomial is
    \[ i(P, t) = \frac{13}{6}t^3 + t^2 - \frac{1}{6}t + 1. \]
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Families of integral polytopes that are known to be Ehrhart positive.

- Standard simplices.
- Zonotopes.
- Stanley-Pitman polytopes.
- Cyclic polytopes.
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**Conjecture 2** (DeLoera-Haws-Koeppe). *All matroid polytopes are Ehrhart positive.*

We consider *generalized permutohedra*, a family of polytopes that include both Stanley-Pitman polytopes and matroid polytopes.

**Conjecture 3** (Castillo-L.). *All integral generalized permutohedra are Ehrhart positive.*
**Usual permutohedra**

**Definition.** Suppose $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is a (nondecreasing) sequence. We define the *usual permutohedron*

$$\text{Perm}(\mathbf{v}) := \text{conv} \left\{ (v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(n)}) : \sigma \in S_n \right\}.$$  

- If $\mathbf{v} = (1, 2, \cdots, n)$, we get the *regular permutohedron* $\Pi_{n-1}$.

**Example.** $\Pi_2$:

Any usual permutohedron in $\mathbb{R}^n$ is $(n - 1)$-dimensional.
Generalized permutohedra

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**Alternative definition**

Let $V$ be the subspace of $\mathbb{R}^n$ defined by $x_1 + x_2 + \cdots + x_n = 0$. The *braid arrangement fan* denoted by $B_n$, is the complete fan in $V$ given by the hyperplanes

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**Proposition 4** (Postnikov-Reiner-Williams). A polytope $P \in \mathbb{R}^n$ is a generalized permutoheron if and only if its normal fan is refined by the braid arrangement fan $B_n$. 
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Postnikov studied the \textit{y-family}, a subset of generalized permutohedra defined by

$$Py = \sum_{S \subseteq [n]} y_S \Delta_S$$

where

$$\Delta_S = \text{conv}(e_i : i \in S)$$

and the $y_S$ all nonnegative.

He gave an explicit formula for the Ehrhart polynomial of any polytope in this family. As a consequence of his formula, \textit{any polytope in the y-family is Ehrhart positive}. 
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The \textit{y-family} includes: regular permutohedra, associahedra, cyclohedra, Stanley-Pitman polytopes, and more.
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Unfortunately, it fails to contain the matroid polytopes.
PART II:

McMullen’s formula and consequences
McMullen’s formula

**Definition.** Suppose $F$ is a face of $P$. The *feasible cone* of $P$ at $F$, denoted by $\text{fcone}(F, P)$, is the cone of all feasible directions of $P$ at $F$.

The *pointed feasible cone* of $P$ at $F$ is $\text{fcone}^p(F, P) = \text{fcone}(F, P)/L$, where $L$ is the subspace spanned by $F$. In general, $\text{fcone}^p(F, P)$ is $k$-dim pointed cone where if $F$ is codimensional $k$. 
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In 1975 Danilov asked if it is possible to assign values $\Psi(C)$ to all rational cones $C$ such that the following *McMullen’s formula* holds

$$|P \cap \mathbb{Z}^d| = \sum_{F: \text{a face of } P} \alpha(F, P) \text{vol}(F).$$

where $\alpha(F, P) := \Psi(\text{fcone}^p(F, P))$. 
**McMullen’s formula**

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where \( \alpha(F, P) := \Psi(fcone^p(F, P)) \).

- McMullen proved it was possible in a non constructive way.
- Subsequently, explicit constructions of \( \Psi/\alpha(F, P) \) were given by Morelli, Pommersheim-Thomas, and Berline-Vergne.

We will use Berline-Vergne’s construction, which we will refer to as the *BV-construction*. 
An expression for Ehrhart coefficients

Given an integral polytope \( P \subseteq \mathbb{R}^d \), any dilation \( tP \) of \( P \) is integral as well. We have

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i(P, t) = |tP \cap \mathbb{Z}^d| = \sum_{F: \text{a face of } P} \alpha(tF, tP) \text{vol}(tF)
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Hence, the coefficient of $t^k$ in $i(P, t)$ is given by

$$\sum_{F: \text{a } k\text{-dimensional face of } P} \alpha(F, P) \text{vol}(F).$$
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Hence, the coefficient of $t^k$ is positive if $\alpha(F, P)$ is positive for any $k$-dimensional face $F$ of $P$.

Moreover, as long as all $\alpha$ for $P$ are positive, $P$ is Ehrhart positive.
For the rest of this part, we assume that $\alpha$ is the BV-construction.

**Theorem 5** (Castillo-L.). Suppose $\alpha(F, \Pi_{n-1}) > 0$ for any $k$-dimensional face $F$ of the regular permutohedron $\Pi_{n-1}$. Then $\alpha(G, Q) > 0$ for any $k$-dimensional face $G$ of any generalized permutohedron $Q$ in $\mathbb{R}^n$. 
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### Diagram

- **Regular Permutohedron**
- **Usual Permutohedra**
- **Generalized Permutohedra**
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A stronger conjecture

Conjecture 6 (Castillo-L.). *The $\alpha$ values (from the BV-construction) of the regular permutoheron $\Pi_{n-1}$ are all positive.*

This conjecture clearly implies our first conjecture by the reduction theorem.

**Note.** The “regular permutohedron $\Pi_{n-1}$” can be replaced with “any generalized permutohedron whose normal fan is the braid arrangement fan $B_n$”.

Thus we may state this conjecture as “the $\alpha$ values for $B_n$ are all positive”.

A more general form of the reduction theorem

The reduction theorem does not only work for $\Pi_{n-1}$ and generalized permutohedra.

**Theorem 7** (Castillo-L.). *Suppose $Q$ is a deformation of $P$, or the normal fan of $P$ is a refinement of the normal fan of $Q$. If $\alpha(F, P) > 0$ for any $k$-dimensional face $F$ of $P$, then $\alpha(G, Q) > 0$ for any $k$-dimensional face $G$ of $Q$.***
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**Lemma** (Castillo-L.). *The $\alpha$ values for regular permutohedra of dimension $\leq 6$ are all positive.*
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Partial results to the second conjecture:

**Lemma (Castillo-L.).** The $\alpha$ values for regular permutohedra of dimension $\leq 6$ are all positive.

**Lemma (Castillo-L.).** $\alpha(F, \Pi_{n-1}) > 0$ for any face $F$ of $\Pi_{n-1}$ of codimension 2 or 3.
What can we show?

Partial results to the second conjecture:

**Lemma** (Castillo-L.). The $\alpha$ values for regular permutohedra of dimension $\leq 6$ are all positive.

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**Lemma** (Castillo-L.). $\alpha(E, \Pi_{n-1}) > 0$ for any edge $E$ of $\Pi_{n-1}$ of dimension $\leq 100$. 
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Partial results to the second conjecture:

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**Lemma** (Castillo-L.). $\alpha(E, \Pi_{n-1}) > 0$ for any edge $E$ of $\Pi_{n-1}$ of dimension $\leq 100$.

Applying the reduction theorem, we get:

**Corollary** (Castillo-L.). *i. Any integral generalized permutohedron of dimension $\leq 6$ is Ehrhart positive.*

*ii. The third and fourth coefficients in the Ehrhart polynomial of any integral generalized permutohedron is positive.*

*iii. The linear coefficient in the Ehrhart polynomial of any integral generalized permutohedron of dimension $\leq 100$ is positive.*
PART III:

The BV-construction and idea of proofs
Computing the BV-construction

In order to attack our conjectures, we want to compute $\alpha$ for cones arising from regular permutohedra $\Pi_{n-1}$, or equivalently compute $\Psi$ arising from the braid arrangement fan $B_n$. 
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In general, the computation of $\Psi(C')$ is quite complicated. However, when the cone $C'$ is unimodular, computations are greatly simplified.
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In general, the computation of $\Psi(C')$ is quite complicated. However, when the cone $C'$ is unimodular, computations are greatly simplified.

**Lemma 8.** Let $C'$ be a one-dimensional (unimodular) cone. Then $\Psi(C') = 1/2$. 
Computing the BV-Construction (cont’d)
Lemma 9. If $C = \text{Cone}(u_1, u_2)$, where $\{u_1, u_2\}$ is a basis for the lattice $\text{span}(u_1, u_2) \cap \mathbb{Z}^n$, then

$$\Psi(C) = \frac{1}{4} + \frac{1}{12} \left( \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_2 \rangle}{\langle u_2, u_2 \rangle} \right).$$
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Example. Consider the polygon $P$ in $\mathbb{R}^2$ with vertices $v_1 = (0, 0)$, $v_2 = (2, 0)$, and $v_3 = (0, 1)$. Let $C_i = f\text{cone}^p(v_i, P)$. 
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$C_1 = \text{Cone}((1, 0), (0, 1))$ is a unimodular cone. Thus,

$$
\alpha(v_1, P) = \Psi(C_1) = \frac{1}{4} + \frac{1}{12} \left( \frac{0}{1} + \frac{0}{1} \right) = \frac{1}{4}.
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Lemma 9. If $C = \text{Cone}(u_1, u_2)$, where $\{u_1, u_2\}$ is a basis for the lattice $\text{span}(u_1, u_2) \cap \mathbb{Z}^n$, then

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$$\alpha(v_1, P) = \Psi(C_1) = \frac{1}{4} + \frac{1}{12} \left( \frac{0}{1} + \frac{0}{1} \right) = \frac{1}{4}.$$

$C_2 = \text{Cone}((-2, 1), (-1, 0))$ is a unimodular cone. Thus,

$$\alpha(v_2, P) = \Psi(C_2) = \frac{1}{4} + \frac{1}{12} \left( \frac{2}{5} + \frac{2}{1} \right) = \frac{9}{20}.$$
Computing the BV-Construction (cont’d)

\[ C_3 = \text{Cone}((0, -1), (2, -1)), \text{ which is not unimodular.} \]  
So we cannot directly apply the formula to compute \( \Psi(C_3) \).
Computing the BV-Construction (cont’d)

\[ C_3 = \text{Cone}((0, -1), (2, -1)), \text{ which is not unimodular. So we cannot directly apply the formula to compute } \Psi(C_3). \text{ In order to compute it, we first decompose } C_3: \]

\[ [C_3] = [\text{Cone } ((0, -1), (1, -1))] + [\text{Cone } ((1, -1), (2, -1))] - [\text{Cone } ((1, -1))]. \]
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\]

We apply the formula to the two first cones in the above decomposition and get \( \Psi \) values of \( \frac{3}{8} \) and \( \frac{17}{40} \). Since the last cone is one-dimensional, we get its \( \Psi \) value to be \( \frac{1}{2} \). 

![Diagram with vectors](image)
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\begin{align*}
[C_3] &= \left[ \text{Cone} ((0, -1), (1, -1)) \right] + \left[ \text{Cone} ((1, -1), (2, -1)) \right] - \left[ \text{Cone} ((1, -1)) \right].
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We apply the formula to the two first cones in the above decomposition and get \( \Psi \) values of \( \frac{3}{8} \) and \( \frac{17}{40} \). Since the last cone is one-dimensional, we get its \( \Psi \) value to be \( \frac{1}{2} \). Finally, by \( \Psi \) is a valuation function, we get

\[
\alpha(v_3, P) = \Psi(C_3) = \frac{3}{8} + \frac{17}{40} - \frac{1}{2} = \frac{3}{10}.
\]
Lemma 10. If $C = \text{Cone}(u_1, u_2, u_3)$ where $u_1, u_2, u_3$ is a basis for the lattice $\text{span}(u_1, u_2) \cap \mathbb{Z}^n$, then

$$\Psi(C) = \frac{1}{8} + \frac{1}{24} \left( \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_2 \rangle}{\langle u_2, u_2 \rangle} + \frac{\langle u_1, u_3 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_3 \rangle}{\langle u_3, u_3 \rangle} + \frac{\langle u_3, u_2 \rangle}{\langle u_2, u_2 \rangle} + \frac{\langle u_3, u_2 \rangle}{\langle u_3, u_3 \rangle} \right).$$
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Remark 11. The formulas for 2-dim and 3-dim unimodular cones appear to be simple. However, the apparent simplicity breaks down for dimension 4. The formula for 4-dim unimodular cones include (way) more than 1000 terms.
Computing the BV-Construction (cont’d)

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\[
\Psi(C') = \frac{1}{8} + \frac{1}{24} \left( \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_2 \rangle}{\langle u_2, u_2 \rangle} + \frac{\langle u_1, u_3 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_3 \rangle}{\langle u_3, u_3 \rangle} + \frac{\langle u_3, u_2 \rangle}{\langle u_2, u_2 \rangle} + \frac{\langle u_3, u_2 \rangle}{\langle u_3, u_3 \rangle} \right).
\]

Remark 11. The formulas for 2-dim and 3-dim unimodular cones appear to be simple. However, the apparent simplicity breaks down for dimension 4. The formula for 4-dim unimodular cones include (way) more than 1000 terms.

Fact 12. \( \Psi \) is computed recursively. So lower dimensional cones are easier to compute.

Recall that if \( F \) is a codimension \( k \) face of \( P \), then \( f_{\text{cone}^p}(F, P) \) is \( k \)-dimensional. Thus, \( \alpha(F, P) \) is easier to compute if \( F \) is a higher dimensional face.
Proofs of lemmas

Lemma (Castillo-L.). *The $\alpha$ values for regular permutohedra of dimension $\leq 6$ are all positive.*

*Proof.* Directly compute all the $\alpha$'s. □
Ehrhart positivity for generalized permutohedra

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Lemma (Castillo-L.). $\alpha(F, \Pi_{n-1}) > 0$ for any face $F$ of $\Pi_{n-1}$ of codimension 2 or 3.

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Lemma 13 (Castillo-L.). $\alpha(E, \Pi_{n-1}) > 0$ for any edge $E$ of $\Pi_{n-1}$ of dimension $\leq 100$.

The approaches used above do not work. Since $\alpha(E, \Pi_{n-1})$ is $\Psi$ of an $(n - 2)$-dimensional cone, which is very hard to compute directly.
Proofs of lemmas

**Lemma (Castillo-L.).** The α values for regular permutohedra of dimension \( \leq 6 \) are all positive.

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*Proof.* We have precise formulas for \( \Psi \) of unimodular cones of dimension \( \leq 3 \). Applying these to regular permutohedra, we get \( \alpha \)-positivity for faces of codimension \( \leq 3 \). □

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**Remark.** The number “100” in the lemma can be pushed further.
The symmetry property

**Lemma.** The valuation $\Psi$ (from the BV-construction) is symmetric about the coordinates, i.e., for any cone $C' \in \mathbb{R}^n$ and any signed permutation $(\sigma, s) \in \mathfrak{S}_n \times \{\pm 1\}^n$, we have

$$
\Psi(C') = \Psi((\sigma, s)(C')),
$$

where $(\sigma, s)(C') = \{s_1 x_{\sigma(1)}, s_2 x_{\sigma(2)}, \ldots, s_n x_{\sigma(n)} : (x_1, \ldots, x_n) \in C\}$. 

Recall that the coefficient of $t^k$ in $i(P, t)$ is given by

$$\sum_{F: \text{a } k\text{-dimensional face of } P} \alpha(F, P) \text{ vol}(F).$$

In particular, the coefficient of the linear term is given by

$$\sum_{E: \text{edge of } P} \alpha(E, P) \text{ vol}(E).$$
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**General idea:** Suppose you have a family of polytopes such that

- they have same pointed feasible cones (for edges) up to signed permutations, and thus have the same $\alpha$-values;
- the Ehrhart polynomial of each polytope in the family is known (or at least the linear Ehrhart coefficient is known).

Then as long as you have enough “independent” polytopes in your family, you can figure out the $\alpha$-values.
Example. When $n = 3 : \Pi_2 = \text{Perm}((1, 2, 3)) = \text{conv}\{\sigma : \sigma \in \mathfrak{S}_3\}$.

The pointed feasible cones of the six edges of $\Pi_2$ are

$$Cone((1, 1, -2)), \quad Cone((2, -1, -1)), \quad Cone((1, -2, 1)),$$
$$Cone((-1, -1, 2)), \quad Cone((-2, 1, 1)), \quad Cone((-1, 2, -1)),$$
Example. When \( n = 3 \) : \( \Pi_2 = \text{Perm}((1, 2, 3)) = \text{conv}\{\sigma : \sigma \in \mathcal{S}_3\} \).

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By the symmetry property of \( \Psi \), these cones all have the same value. Therefore, all \( \alpha(E, \Pi_2) \) are a single value, say \( \alpha \).
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By the symmetry property of \( \Psi \), these cones all have the same value. Therefore, all \( \alpha(E, \Pi_2) \) are a single value, say \( \alpha \).

The Ehrhart polynomial of \( \Pi_2 \) is \( 3t^2 + 3t + 1 \). Thus,
\[
3 = \sum_{E} \alpha(E, \Pi_2) \cdot \text{vol}(E) = 6\alpha \quad \Rightarrow \quad \alpha = 1/2 > 0.
\]
Idea of the proof of Lemma 13 (cont’d)

Example. When $n = 4$: $\Pi_3 = \text{Perm}((1, 2, 3, 4)) = \{\sigma : \sigma \in \mathcal{S}_4\}$. 
Idea of the proof of Lemma 13 (cont’d)

Example. When \( n = 4 \) : \( \Pi_3 = \text{Perm}((1, 2, 3, 4)) = \{ \sigma : \sigma \in S_4 \} \).

\( \Pi_3 \) have 36 edges of two kinds. 24 short edges have the same \( \alpha \)-values, say \( \alpha_1 \), and 12 long edges have the same \( \alpha \)-values, say \( \alpha_2 \).
Example. When $n = 4 : \Pi_3 = \text{Perm}((1, 2, 3, 4)) = \{ \sigma : \sigma \in S_4 \}.$

$\Pi_3$ have 36 edges of two kinds. 24 short edges have the same $\alpha$-values, say $\alpha_1$, and 12 long edges have the same $\alpha$-values, say $\alpha_2$.

The Ehrhart polynomial of $\Pi_3$ is $16t^3 + 15t^2 + 6t + 1$. Thus,

$$6 = \sum_E \alpha(E, \Pi_3) \cdot \text{vol}(E) = 24\alpha_1 + 12\alpha_2.$$
**Idea of the proof of Lemma 13 (cont’d)**

**Example.** When $n = 4 : \Pi_3 = \text{Perm}((1, 2, 3, 4)) = \{\sigma : \sigma \in S_4\}$.  

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*Not enough equations!*
Idea of the proof of Lemma 13 (cont’d)

Consider the hypersimplex $\Delta_{2,4} = \text{Perm}((0, 0, 1, 1))$. It has 12 edges whose corresponding pointed feasible cones are the same as that of the 12 long edges of $\Pi_3$. So they all have $\alpha$-values $\alpha_2$. 
Idea of the proof of Lemma 13 (cont’d)

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The Ehrhart polynomial of \( \Delta_{2,4} \) is \( \frac{2}{3}t^3 + 2t^2 + \frac{7}{3}t + 1 \). Thus,

\[
\frac{7}{3} = \sum_{E} \alpha(E, \Delta_{2,4}) \cdot \text{vol}(E) = 12\alpha_2.
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Therefore, we solve the $2 \times 2$ linear system, and get

$$\alpha_1 = \frac{11}{72} > 0, \quad \alpha_2 = \frac{7}{36} > 0.$$
Idea of the proof of Lemma 13 (cont’d)

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For arbitrary $n$: The linear Ehrhart coefficient of some polytopes in the $y$-family can be easily described. Using these, we were able to set up an explicit triangular linear system for $\{\alpha(E, \Pi_{n-1}) : E \text{ is an edge of } \Pi_{n-1}\}$ for any $n$. 
PART IV:

Other questions and results
The solution $\Psi$ to McMullen’s formula is not unique since we know there are different constructions.

*Observation* 14. When we prove Lemma 13, we did not really compute Berline-Vergne’s construction. Instead, we just use the fact that their construction is symmetric about the coordinates to set up linear system to solve.

E.g., in the case of $\Pi_3$ we did in the last example, as long as we know a construction $\Psi$

- satisfies McMullen’s formula, and
- is symmetric about the coordinates,

we will set up exactly the same $2 \times 2$ linear system, and find exactly the same two $\alpha$-values.

So $\Psi$ of the cones appeared in the example or the values of $\alpha(E, \Pi_3)$ are unique.
Question 15. Is it true that $\Psi$ in McMullen’s formula is uniquely determined if we require it to be symmetric about the coordinates?
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If so, then the BV-construction is the only symmetric construction.
**Uniqueness of the construction of $\Psi$**

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If so, then the BV-construction is the only symmetric construction.

**Theorem 16** (Castillo-L.). *Suppose $\Psi$ is a solution to McMullen’s formula and is symmetric about the coordinates. Then the values of $\Psi$ on cones arising from generalized permutohedron are uniquely determined.*
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If so, then the BV-construction is the only symmetric construction.

Theorem 16 (Castillo-L.). Suppose $\Psi$ is a solution to McMullen's formula and is symmetric about the coordinates. Then the values of $\Psi$ on cones arising from generalized permutohedron are uniquely determined.

Idea of proof: Use mixed Ehrhart theory.
Mixed Ehrhart Theorem

Consider the following Minkowski sum:

\[ P = w_1 P_1 + w_2 P_2 + \cdots + w_k P_k, \]

where \( w_i \) are variables and \( P_i \) are polytopes.

**Mixed Ehrhart Theorem** The number of integer points in \( P \) is a polynomial in \( w_i \)'s. The coefficients are called *mixed Ehrhart coefficients.*
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Mixed Ehrhart Theorem The number of integer points in \( P \) is a polynomial in \( w_i \)'s. The coefficients are called mixed Ehrhart coefficients.

Postnikov showed that usual permutohedra are Minkowski sums of hypersimplices.

\[ \text{Perm}(v) = w_1 \Delta_{1,n} + w_2 \Delta_{2,n} + \cdots + w_{n-1} \Delta_{n-1,n}, \]

where

\[ w_i := v_{i+1} - v_i \text{ for } i = 1, 2, \ldots, n - 1, \]

and the hypersimplex \( \Delta_{k,n} \) is defined as

\[ \Delta_{k,n} = \text{Perm}(\underbrace{0, \cdots, 0}_{n-k}, \underbrace{1, \cdots, 1}_{k}). \]
Theorem 17 (Castillo-L.). Suppose $\Psi$ is a solution to McMullen’s formula and is symmetric about the coordinates. Then the $\alpha$ values for the regular permutohedron $\Pi_{n-1}$ (or the braid arrangement fan $B_n$) are positive scalars of mixed Ehrhart coefficients of hypersimplices.
Consequences

i. We obtain a proof for Theorem 16 (the theorem on uniqueness of $\Psi$).
Consequences

i. We obtain a proof for Theorem 16 (the theorem on uniqueness of $Ψ$).

ii. The following three statements are equivalent:

(a) All $α$ values of $Π_{n-1}$ are positive. (The strong conjecture).

(b) All mixed Ehrhart coefficients of hypersimplices are positive.

(c) Let $X$ be corresponding toric variety to the braid arrangement fan $B_n$. The Todd class is positive with respect to the torus invariant cycles, that is

$$\text{Todd}(X) = \sum_{\sigma \in B_n} r_\sigma[V(\sigma)],$$

for some $r_\sigma > 0$. 