Higher integrality conditions and volumes of slices

Fu Liu

University of California, Davis

Combinatorics Seminar

University of California, Berkeley

April 5, 2010

Outline

- Basic definitions and theory of Ehrhart polynomials
- Motivation: Ehrhart polynomials of cyclic polytopes
- Main results

Basic definitions

A *(convex) polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \mathrm{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

Basic definitions

A *(convex) polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \mathrm{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

Throughout this talk, we assume P is *full-dimensional*, i.e., $\dim(P) = d$.

Basic definitions

A *(convex) polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \mathrm{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

Throughout this talk, we assume P is *full-dimensional*, i.e., $\dim(P) = d$.

The d-dimensional *lattice* $\mathbb{Z}^d \subset \mathbb{R}^d$ is the collection of all points with integer coordinates. Any point in the lattice is called a *lattice point* or an *integral point*.

Basic definitions

A *(convex) polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \mathrm{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

Throughout this talk, we assume P is *full-dimensional*, i.e., $\dim(P) = d$.

The d-dimensional *lattice* $\mathbb{Z}^d \subset \mathbb{R}^d$ is the collection of all points with integer coordinates. Any point in the lattice is called a *lattice point* or an *integral point*.

A polytope P is *integral* if its vertices are all lattice points.

Lattice points of a polytope

Lattice points of a polytope

Definition 1. For any polytope $P \subset \mathbb{R}^d$ and positive integer $m \in \mathbb{N}$, the *mth dilated polytope* of P is $mP = \{m\mathbf{x} \ : \ \mathbf{x} \in P\}$. We denote by

$$i(P, m) = |mP \cap \mathbb{Z}^d|$$

the number of lattice points in mP.

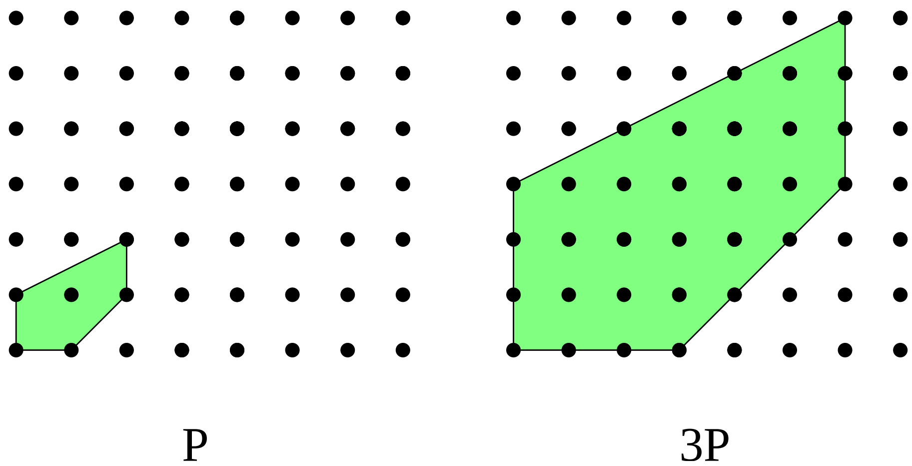
Lattice points of a polytope

Definition 1. For any polytope $P \subset \mathbb{R}^d$ and positive integer $m \in \mathbb{N}$, the *mth dilated polytope* of P is $mP = \{m\mathbf{x} \ : \ \mathbf{x} \in P\}$. We denote by

$$i(P, m) = |mP \cap \mathbb{Z}^d|$$

the number of lattice points in mP.

Example:



Examples of integral polytopes

Examples of integral polytopes

(i) When $d = 1$, P is an interval $[a, b]$, where $a, b \in \mathbb{Z}$. Then $mP = [ma, mb]$ and

$$i(P, m) = (b - a)m + 1.$$

Examples of integral polytopes

(i) When $d = 1$, P is an interval $[a, b]$, where $a, b \in \mathbb{Z}$. Then $mP = [ma, mb]$ and

$$i(P, m) = (b - a)m + 1.$$

(ii) When $d = 2$, P is an integral polygon, and so is mP. Pick's theorem states that for any integral polygon Q:

$$\mathrm{area}(Q) = |Q \cap \mathbb{Z}^2| - \frac{1}{2}|\partial(Q) \cap \mathbb{Z}^2| - 1.$$

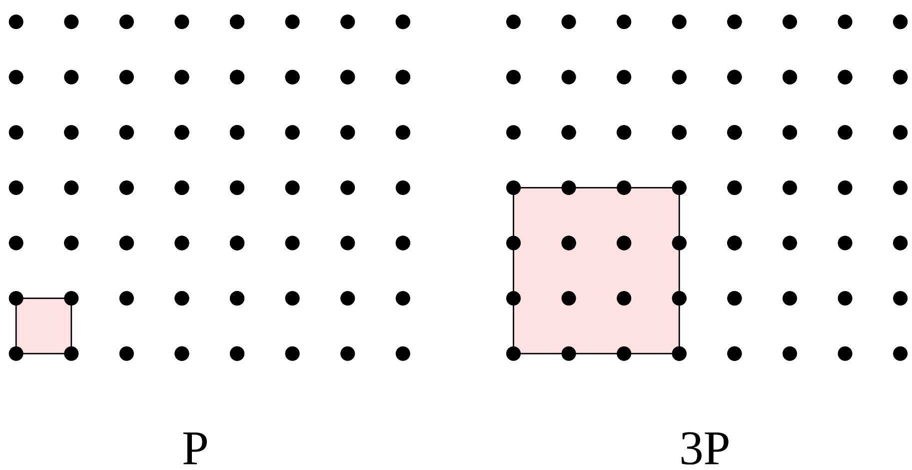
Hence,

$$\begin{aligned} i(P, m) &= \mathrm{area}(mP) + \frac{1}{2}|\partial(mP) \cap \mathbb{Z}^2| + 1 \\ &= \mathrm{area}(P)m^2 + \frac{1}{2}|\partial(P) \cap \mathbb{Z}^2|m + 1 \end{aligned}$$

Examples of integral polytopes

(iii) For any d, let P be the convex hull of the set $\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1\}$, i.e. P is the *unit cube* in $\mathbb{R}^d$. Then it is obvious that

$$i(P, m) = (m + 1)^d.$$



$$i(P, 3) = (3 + 1)^2$$

Theorem of Ehrhart (on integral polytopes)

Theorem 2 (Ehrhart)**.** *Let P be a d-dimensional integral polytope. Then $i(P, m)$ is a polynomial in m of degree d.*

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of P.

Coefficients of Ehrhart polynomials

If P is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, m)$?

Coefficients of Ehrhart polynomials

If P is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, m)$?

➠ The leading coefficient of $i(P, m)$ is the volume $\mathrm{Vol}(P)$ of P.

Coefficients of Ehrhart polynomials

If P is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, m)$?

- The leading coefficient of $i(P, m)$ is the volume $\mathrm{Vol}(P)$ of P.
- The second coefficient equals $1/2$ of the sum of the normalized volumes of each facet.

Coefficients of Ehrhart polynomials

If P is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P,m)$?

- The leading coefficient of $i(P,m)$ is the volume $\mathrm{Vol}(P)$ of P.
- The second coefficient equals $1/2$ of the sum of the normalized volumes of each facet.
- The constant term of $i(P,m)$ is always 1.

Coefficients of Ehrhart polynomials

If P is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, m)$?

- The leading coefficient of $i(P, m)$ is the volume $\mathrm{Vol}(P)$ of P.
- The second coefficient equals $1/2$ of the sum of the normalized volumes of each facet.
- The constant term of $i(P, m)$ is always 1.
- No simple forms known for other coefficients for general polytopes.

Coefficients of Ehrhart polynomials

If P is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, m)$?

- The leading coefficient of $i(P, m)$ is the volume $\mathrm{Vol}(P)$ of P.
- The second coefficient equals $1/2$ of the sum of the normalized volumes of each facet.
- The constant term of $i(P, m)$ is always 1.
- No simple forms known for other coefficients for general polytopes.
  - It is NOT even true that all the coefficients are nonnegative. For example, for the polytope P with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,13)$, its Ehrhart polynomial is

$$i(P, n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$

Questions

- When can we have simple forms for all coefficients?

Questions

- When can we have simple forms for all coefficients?
- When are they positive?

Questions

- When can we have simple forms for all coefficients?
- When are they positive?
- When can the coefficients be described by volumes?

Cyclic polytope

Beck, De Loera, Develin, Pfeifle and Stanley conjectured that the Ehrhart polynomial of an integral cyclic polytope has a simple formula.

Recall that given $n > d$, and $T = \{t_1 < \cdots < t_n\}$, a d-dimensional *cyclic polytope* $C_d(T) = C_d(t_1, \ldots, t_n)$ is the convex hull $\mathrm{conv}\{v_d(t_1), v_d(t_2), \ldots, v_d(t_n)\}$ of the n distinct points $\nu_d(t_i), 1 \le i \le n$, on the moment curve.

The *moment curve* (also known as *rational normal curve*) in $\mathbb{R}^d$ is defined by

$$\nu_d : \mathbb{R} \to \mathbb{R}^d, t \mapsto \nu_d(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix}.$$

Example: $T = \{1, 2, 3, 4\}, d = 3$:

$C_d(T)$ is the convex polytope whose vertices are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}.$

Theorem 3. *Suppose $P = C_d(T)$ is a d-dimensional integral cyclic polytope. Then*

$$i(P, m) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P)) m^i.$$

where $\pi^{(d-i)} : \mathbb{R}^d \longrightarrow \mathbb{R}^i$ is the projection which drops the last $d - i$ coordinates.

Example: $T = \{1, 2, 3, 4\}, d = 3$:

Example: $T = \{1, 2, 3, 4\}, d = 3:$

➠ $$P = C_d(T) = \mathrm{conv}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix} \right\} : \mathrm{Vol}(P) = 2.$$

Example: $T = \{1,2,3,4\}, d = 3:$

➠ $P = C_d(T) = \text{conv}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}\} : \text{Vol}(P) = 2.$

➠ $\pi^{(1)}(P) = \text{conv}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \end{pmatrix}\} : \text{Vol}(\pi^{(1)}(P)) = 4.$

Example: $T = \{1, 2, 3, 4\}, d = 3:$

➠ $P = C_d(T) = \text{conv}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}\} : \text{Vol}(P) = 2.$

➠ $\pi^{(1)}(P) = \text{conv}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \end{pmatrix}\} : \text{Vol}(\pi^{(1)}(P)) = 4.$

➠ $\pi^{(2)}(P) = \text{conv}\{1, 2, 3, 4\} = [1, 4] : \text{Vol}(\pi^{(2)}(P)) = 3.$

Example: $T = \{1, 2, 3, 4\}, d = 3:$

➠ $P = C_d(T) = \text{conv}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}\}$: $\text{Vol}(P) = 2.$

➠ $\pi^{(1)}(P) = \text{conv}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \end{pmatrix}\}$: $\text{Vol}(\pi^{(1)}(P)) = 4.$

➠ $\pi^{(2)}(P) = \text{conv}\{1, 2, 3, 4\} = [1, 4]$: $\text{Vol}(\pi^{(2)}(P)) = 3.$

➠ $\pi^{(3)}(P) = \mathbb{R}^0$: $\text{Vol}(\pi^{(3)}(P)) = 1.$

Example: $T = \{1, 2, 3, 4\}, d = 3$:

- $P = C_d(T) = \text{conv}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}\}$: $\text{Vol}(P) = 2.$
- $\pi^{(1)}(P) = \text{conv}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \end{pmatrix}\}$: $\text{Vol}(\pi^{(1)}(P)) = 4.$
- $\pi^{(2)}(P) = \text{conv}\{1, 2, 3, 4\} = [1, 4]$: $\text{Vol}(\pi^{(2)}(P)) = 3.$
- $\pi^{(3)}(P) = \mathbb{R}^0$: $\text{Vol}(\pi^{(3)}(P)) = 1.$
- The theorem says:

$$i(P, m) = 2m^3 + 4m^2 + 3m + 1.$$

Question:

Are there other integral polytopes whose Ehrhart polynomials have the same form as cyclic polytopes?

In other words, what kind of integral polytopes P are there whose Ehrhart polynomials have the form

$$i(P,m) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P)) m^i?$$

Higher integrality conditions for affine spaces

Definition 4. An ℓ-dimensional affine space $U \subset \mathbb{R}^d$ is *integral* if

$$\pi^{(d-\ell)}(U \cap \mathbb{Z}^d) = \mathbb{Z}^\ell.$$

Or equivalently, the projection $\pi^{(d-\ell)}$ induces a bijection between $U \cap \mathbb{Z}^d$ and $\mathbb{Z}^\ell$.

Higher integrality conditions for affine spaces

Definition 4. An ℓ-dimensional affine space $U \subset \mathbb{R}^d$ is *integral* if

$$\pi^{(d-\ell)}(U \cap \mathbb{Z}^d) = \mathbb{Z}^\ell.$$

Or equivalently, the projection $\pi^{(d-\ell)}$ induces a bijection between $U \cap \mathbb{Z}^d$ and $\mathbb{Z}^\ell$.

Example:

(i) $\ell = 0$: U is integral if and only if U is a lattice point.

Higher integrality conditions for affine spaces

Definition 4. An ℓ-dimensional affine space $U \subset \mathbb{R}^d$ is *integral* if

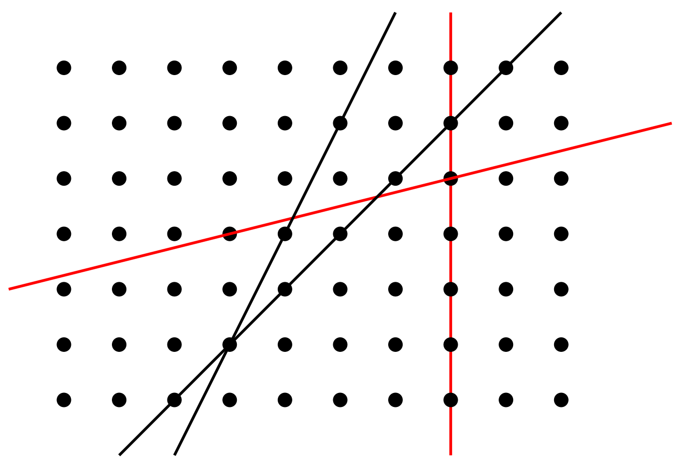
$$\pi^{(d-\ell)}(U \cap \mathbb{Z}^d) = \mathbb{Z}^\ell.$$

Or equivalently, the projection $\pi^{(d-\ell)}$ induces a bijection between $U \cap \mathbb{Z}^d$ and $\mathbb{Z}^\ell$.

Example:

(i) $\ell = 0$: U is integral if and only if U is a lattice point.

(ii) $\ell = 1$: In $\mathbb{R}^2$,



Higher integrality conditions for affine spaces

Definition 4. An ℓ-dimensional affine space $U \subset \mathbb{R}^d$ is *integral* if

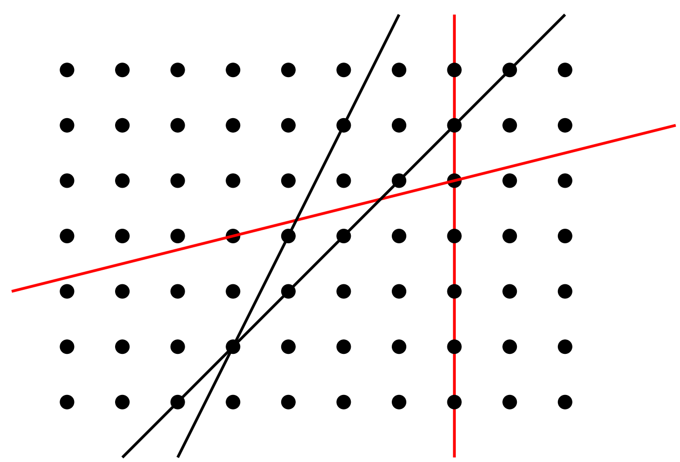
$$\pi^{(d-\ell)}(U \cap \mathbb{Z}^d) = \mathbb{Z}^\ell.$$

Or equivalently, the projection $\pi^{(d-\ell)}$ induces a bijection between $U \cap \mathbb{Z}^d$ and $\mathbb{Z}^\ell$.

Example:

(i) $\ell = 0$: U is integral if and only if U is a lattice point.

(ii) $\ell = 1$: In $\mathbb{R}^2$,



In general, U is integral if and only if U contains a lattice point and $\mathrm{dir}(U) = (1, z_2, \ldots, z_d) \in \mathbb{Z}^d$.

A property of integral cyclic polytopes

For any integral cyclic polytope P, we have that

any affine space determined by a subset of $\mathrm{Vert}(P)$ *is integral.* $(\star)$

A property of integral cyclic polytopes

For any integral cyclic polytope P, we have that

any affine space determined by a subset of $\mathrm{Vert}(P)$ *is integral.* (⋆)

Theorem 5. *Suppose* P *is a polytope satisfing* $(\star)$. *Then*

$$i(P, m) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P)) m^i.$$

A property of integral cyclic polytopes

For any integral cyclic polytope P, we have that

any affine space determined by a subset of $\mathrm{Vert}(P)$ *is integral.* $(\star)$

Theorem 5. *Suppose* P *is a polytope satisfing* $(\star)$. *Then*

$$i(P,m) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P))m^i.$$

Question:

(i) Can we relax $(\star)$ to the following condition?

Any affine space $\mathrm{aff}(F)$ *determined by a face* F *of* P *is integral.* $(\triangle)$

A property of integral cyclic polytopes

For any integral cyclic polytope P, we have that

any affine space determined by a subset of $\mathrm{Vert}(P)$ *is integral.* (⋆)

Theorem 5. *Suppose* P *is a polytope satisfing* $(\star)$*. Then*

$$i(P,m) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P))m^i.$$

Question:

(i) Can we relax $(\star)$ to the following condition?

Any affine space $\mathrm{aff}(F)$ *determined by a face* F *of* P *is integral.* (△)

(ii) What if $(\star)$ or $(\triangle)$ is only satisfied for affine spaces of dimension in a subset $S \subseteq \{0, 1, \ldots, d\}$?

Higher integrality conditions for polytopes

Definition 6. A polytope P is *k-integral* if for any $0 \le \ell \le k$, we have that $\operatorname{aff}(F)$ is integral for any ℓ-dimensional face F of P.

We say P is *fully integral* if $k = d$.

Higher integrality conditions for polytopes

Definition 6. A polytope P is *k-integral* if for any $0 \leq \ell \leq k$, we have that $\mathrm{aff}(F)$ is integral for any ℓ-dimensional face F of P.

We say P is *fully integral* if $k = d$.

Example:

(i) $k = 0 :$ P is 0-integral if and only if P is integral.

Higher integrality conditions for polytopes

Definition 6. A polytope P is *k-integral* if for any $0 \leq \ell \leq k$, we have that $\mathrm{aff}(F)$ is integral for any ℓ-dimensional face F of P.

We say P is *fully integral* if $k = d$.

Example:

(i) $k = 0 :$ P is 0-integral if and only if P is integral.

(ii) $k = 1 :$ P is 1-integral if and only if P is integral and $\mathrm{dir}(e) = (1, z_2, \dots, z_d) \in \mathbb{Z}^d$ for any edge e of P.

Higher integrality conditions for polytopes

Definition 6. A polytope P is *k-integral* if for any $0 \leq \ell \leq k$, we have that $\mathrm{aff}(F)$ is integral for any ℓ-dimensional face F of P.

We say P is *fully integral* if $k = d$.

Example:

(i) $k = 0 :$ P is 0-integral if and only if P is integral.

(ii) $k = 1 :$ P is 1-integral if and only if P is integral and $\mathrm{dir}(e) = (1, z_2, \dots, z_d) \in \mathbb{Z}^d$ for any edge e of P.

Conjecture 7. *If P is k-integral, then for $0 \leq \ell \leq k$, the coefficient of m^ℓ in $i(P, m)$ is* $\mathrm{Vol}(\pi^{d-\ell}(P))$.

Higher integrality conditions for polytopes

Definition 6. A polytope P is *k-integral* if for any $0 \leq \ell \leq k$, we have that $\mathrm{aff}(F)$ is integral for any ℓ-dimensional face F of P.

We say P is *fully integral* if $k = d$.

Example:

(i) $k = 0 : P$ is 0-integral if and only if P is integral.

(ii) $k = 1 : P$ is 1-integral if and only if P is integral and $\mathrm{dir}(e) = (1, z_2, \dots, z_d) \in \mathbb{Z}^d$ for any edge e of P.

Conjecture 7. *If P is k-integral, then for $0 \leq \ell \leq k$, the coefficient of m^ℓ in $i(P, m)$ is $\mathrm{Vol}(\pi^{d-\ell}(P))$.*

Example: $P = \mathrm{conv}\{(0,0,0), (4,0,0), (3,6,0), (2,2,2)\}$. One can check that P is 1-integral.

$$i(P, m) = 8m^3 + 10m^2 + 4m + 1,$$

and $\mathrm{Vol}(\pi^{(d-1)}(P)) = \mathrm{Vol}([0,4]) = 4$ and $\mathrm{Vol}(\pi^{(d-0)}(P)) = \mathrm{Vol}(\mathbb{R}^0) = 1$.

Slices of a polytope

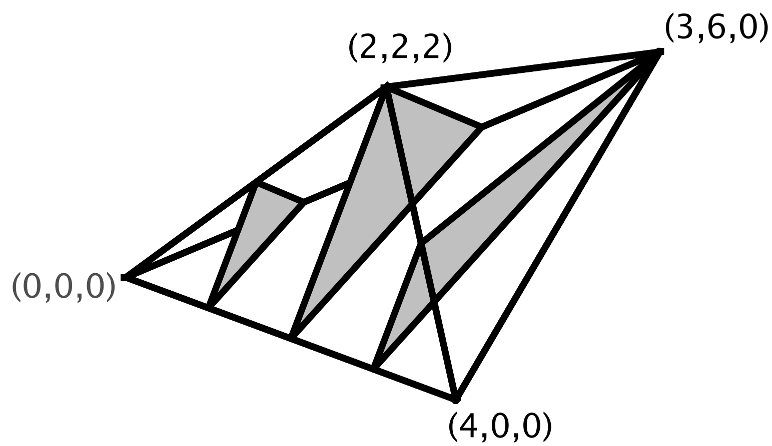
Definition 8. For any $\mathbf{y} \in \pi^{(d-k)}(P)$, we define *the slice of P over $\mathbf{y}$*, denoted by $\pi_{d-k}(\mathbf{y}, P)$, to be the intersection of P with the inverse image of $\mathbf{y}$ under $\pi^{(d-k)}$.

Slices of a polytope

Definition 8. For any $\mathbf{y} \in \pi^{(d-k)}(P)$, we define *the slice of P over $\mathbf{y}$*, denoted by $\pi_{d-k}(\mathbf{y}, P)$, to be the intersection of P with the inverse image of $\mathbf{y}$ under $\pi^{(d-k)}$.

Example: $P = \text{conv}\{(0,0,0),(4,0,0),(3,6,0),(2,2,2)\}$.

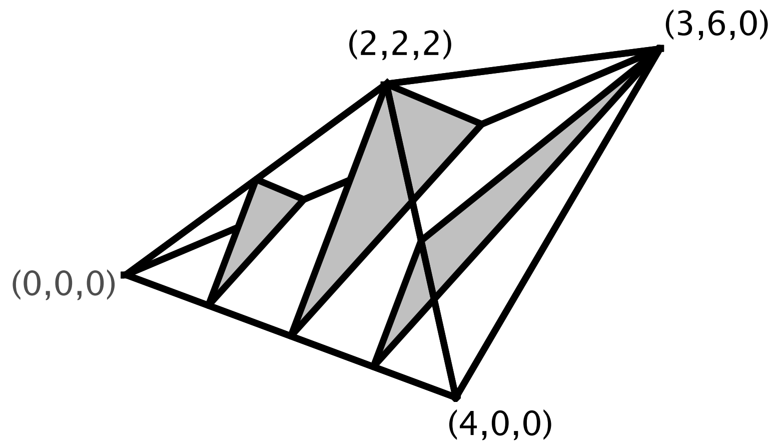
Recall $\pi^{(d-1)}(P) = [0,4]$ and $i(P,m) = 8m^3 + 10m^2 + 4m + 1$.



Slices of a polytope

Definition 8. For any $\mathbf{y} \in \pi^{(d-k)}(P)$, we define *the slice of P over $\mathbf{y}$*, denoted by $\pi_{d-k}(\mathbf{y}, P)$, to be the intersection of P with the inverse image of $\mathbf{y}$ under $\pi^{(d-k)}$.

Example: $P = \text{conv}\{(0,0,0),(4,0,0),(3,6,0),(2,2,2)\}$.
Recall $\pi^{(d-1)}(P) = [0,4]$ and $i(P,m) = 8m^3 + 10m^2 + 4m + 1$.



$i(\pi_2(0,P),m) = 1, i(\pi_2(1,P),m) = m^2+2m+1, i(\pi_2(2,P),m) = 4m^2+4m+1, i(\pi_2(3,P),m) = 3m^2+4m+1$ and $i(\pi_2(4,P),m) = 1$. Their sum is

$$8m^2 + 10m + 5.$$

Main theorems

Theorem 9. *If P is k-integral, then the coefficient of m^ℓ in $i(P, m)$ is*

$$\begin{cases} \mathrm{Vol}(\pi^{d-\ell}(P)) & \text{if } 0 \le \ell \le k, \\ \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{coefficient of } m^{\ell-k} \text{ in } i(\pi_{d-k}(\mathbf{y}, P), m) & \text{if } k+1 \le \ell \le d \end{cases}$$

Main theorems

Theorem 9. *If P is k-integral, then the coefficient of m^ℓ in $i(P, m)$ is*

$$\begin{cases} \mathrm{Vol}(\pi^{d-\ell}(P)) & \text{if } 0 \le \ell \le k, \\ \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{coefficient of } m^{\ell-k} \text{ in } i(\pi_{d-k}(\mathbf{y}, P), m) & \text{if } k+1 \le \ell \le d \end{cases}$$

Theorem 10. *Suppose $k < d$. If P is k-integral, then*

$$\mathrm{Vol}(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)),$$

where Vol_{d-k} is the volume with respect to the lattice $\mathbb{Z}^{d-k}$.

Main theorems

Theorem 9. *If P is k-integral, then the coefficient of m^ℓ in $i(P,m)$ is*

$$\begin{cases} \mathrm{Vol}(\pi^{d-\ell}(P)) & \text{if } 0 \le \ell \le k, \\ \sum_{\mathbf{y}\in\pi^{(d-k)}(P)\cap\mathbb{Z}^k} \text{coefficient of } m^{\ell-k} \text{ in } i(\pi_{d-k}(\mathbf{y},P),m) & \text{if } k+1 \le \ell \le d \end{cases}$$

Theorem 10. *Suppose $k < d$. If P is k-integral, then*

$$\mathrm{Vol}(P) = \sum_{\mathbf{y}\in\pi^{(d-k)}(P)\cap\mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y},P)),$$

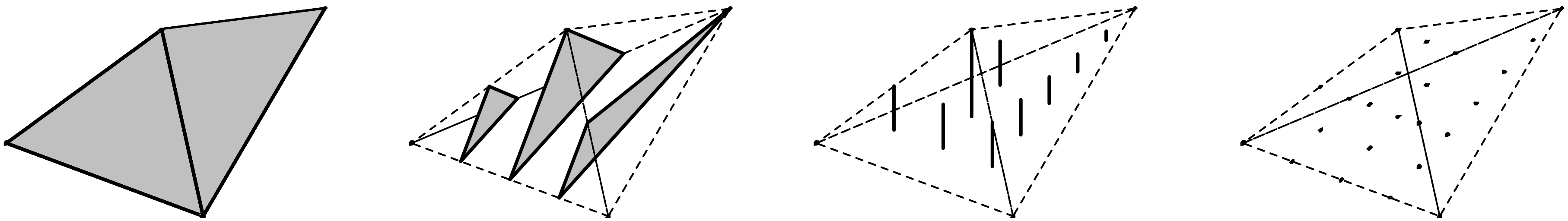
where Vol_{d-k} is the volume with respect to the lattice $\mathbb{Z}^{d-k}$.

Definition 11. We define the *kth S-volume of P* to be

$$\mathrm{SVol}^k(P) = \sum_{\mathbf{y}\in\pi^{(d-k)}(P)\cap\mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y},P)).$$

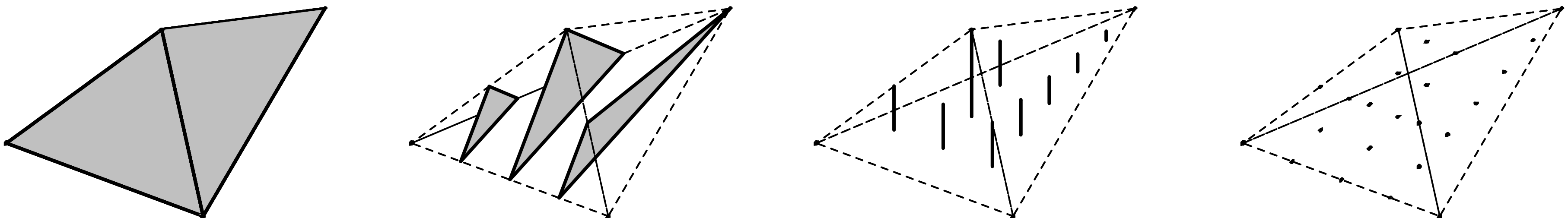
Remarks

Remark 12. $\mathrm{SVol}^0(P) = \mathrm{Vol}(P)$ and $\mathrm{SVol}^d(P) = |P \cap Z^d|$.



Remarks

Remark 12. $\mathrm{SVol}^0(P) = \mathrm{Vol}(P)$ and $\mathrm{SVol}^d(P) = |P \cap Z^d|$.



Remark 13. Theorem 10 says if $k < d$ and P is k-integral, then $\mathrm{Vol}(P) = \mathrm{SVol}^k(P)$. Note that P is ℓ-integral for any $\ell \leq k$, so we have

$$\mathrm{Vol}(P) = \mathrm{SVol}^0(P) = \mathrm{SVol}^1(P) = \cdots = \mathrm{SVol}^k(P).$$

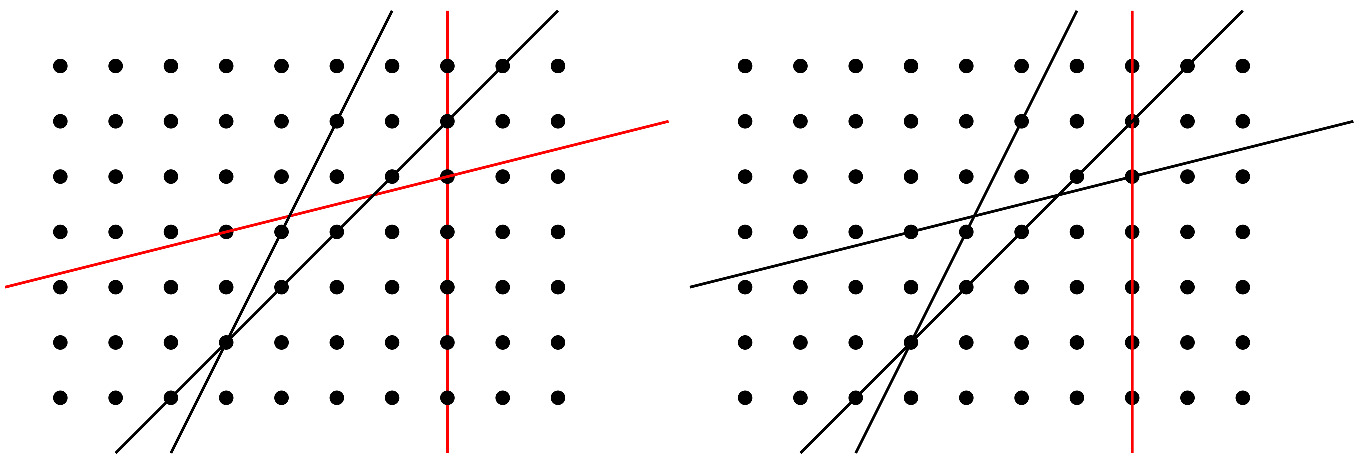
Remarks

Remark 14. The condition k-integral in Theorem 10 can be relaxed to $(k-1)$-integral and *in k-general position*.

Remarks

Remark 14. The condition k-integral in Theorem 10 can be relaxed to $(k-1)$-integral and *in k-general position*.

Example: 1-dimensional affine space in $\mathbb{R}^2$. integral vs in general position:

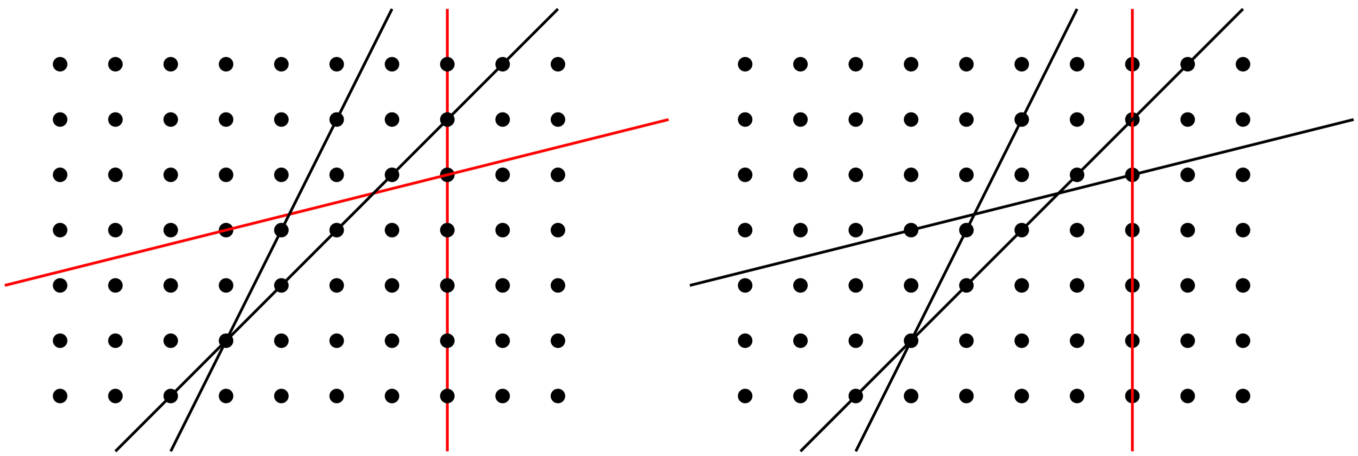


In general, U is in general position if and only if $\mathrm{dir}(U) = (1, y_1, \ldots, y_d) \in \mathbb{R}^d$.

Remarks

Remark 14. The condition k-integral in Theorem 10 can be relaxed to $(k-1)$-integral and *in k-general position*.

Example: 1-dimensional affine space in $\mathbb{R}^2$. integral vs in general position:

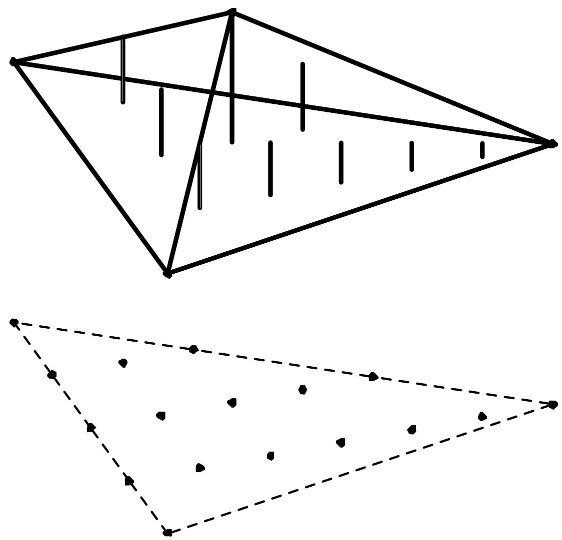


In general, U is in general position if and only if $\mathrm{dir}(U) = (1, y_1, \ldots, y_d) \in \mathbb{R}^d$.

Theorem 9 can be reduced to Theorem 10.

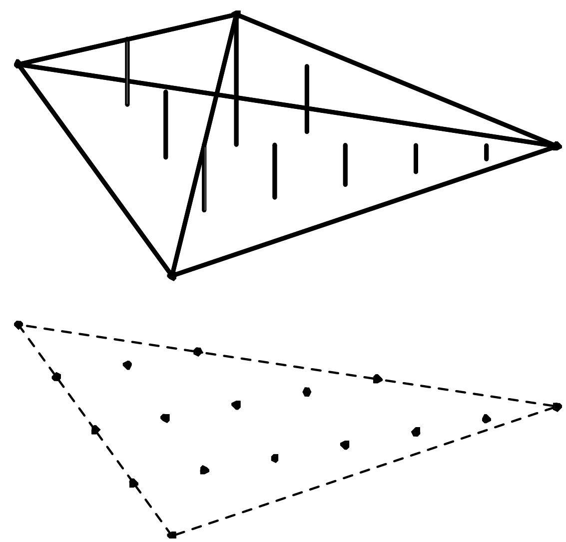
Reduction to volume formula

i. If P is fully integral, then in particular P is $(d-1)$-integral. For any $\mathbf{y} \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}$, the slice $\pi_1(\mathbf{y}, P)$ is either a 1-dimensional integral polytope, or a lattice point. In either case, we have that $|\pi_1(\mathbf{y}, P) \cap \mathbb{Z}^d| = 1 + \mathrm{Vol}_1(\pi_1(\mathbf{y}, P))$.



Reduction to volume formula

i. If P is fully integral, then in particular P is $(d-1)$-integral. For any $\mathbf{y} \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}$, the slice $\pi_1(\mathbf{y}, P)$ is either a 1-dimensional integral polytope, or a lattice point. In either case, we have that $|\pi_1(\mathbf{y}, P) \cap \mathbb{Z}^d| = 1 + \mathrm{Vol}_1(\pi_1(\mathbf{y}, P))$.



Then

$$|P \cap \mathbb{Z}^d| = \sum_{\mathbf{y} \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}} |\pi_1(\mathbf{y}, P) \cap \mathbb{Z}^d|$$

$$= \sum_{\mathbf{y} \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}} (1 + \mathrm{Vol}_1(\pi_1(\mathbf{y}, P))) = |\pi^{(1)}(P) \cap \mathbb{Z}^{d-1}| + \mathrm{Vol}(P).$$

Reduction to volume formula

However, $\pi^{(1)}(P)$ is fully integral. Hence,

$$|P \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P)).$$

Note that P is k-integral $\Rightarrow mP$ is k-integral as well. Therefore,

$$i(P, m) = |mP \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(mP)) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P))m^i.$$

Reduction to volume formula

However, $\pi^{(1)}(P)$ is fully integral. Hence,

$$|P \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P)).$$

Note that P is k-integral $\Rightarrow mP$ is k-integral as well. Therefore,

$$i(P,m) = |mP \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(mP)) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P))m^i.$$

ii. If $k \leq d-1$, the projection $\pi^{(d-k)}(P)$ is fully integral and for any $\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k$, the slice $\pi_{d-k}(\mathbf{y}, P)$ is an integral polytope.

Reduction to volume formula

However, $\pi^{(1)}(P)$ is fully integral. Hence,

$$|P \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P)).$$

Note that P is k-integral $\Rightarrow mP$ is k-integral as well. Therefore,

$$i(P,m) = |mP \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(mP)) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P))m^i.$$

ii. If $k \le d-1$, the projection $\pi^{(d-k)}(P)$ is fully integral and for any $\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k$, the slice $\pi_{d-k}(\mathbf{y}, P)$ is an integral polytope.

We prove Theorem 9 by using the result on fully integral polytopes as well as a local formula relating the number of lattice points to volumes of faces for integral polytopes obtained by Morelli, McMullen, Pommersheim-Thomas, Berline-Vergne.

Sketch of the proof of Theorem 10

Recall the theorem: If P is k-integral, then

$$\mathrm{Vol}(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)) = \mathrm{SVol}^k(P).$$

Sketch of the proof of Theorem 10

Recall the theorem: If P is k-integral, then

$$\mathrm{Vol}(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)) = \mathrm{SVol}^k(P).$$

i. We reduce the problem to the case of $k = 1$.

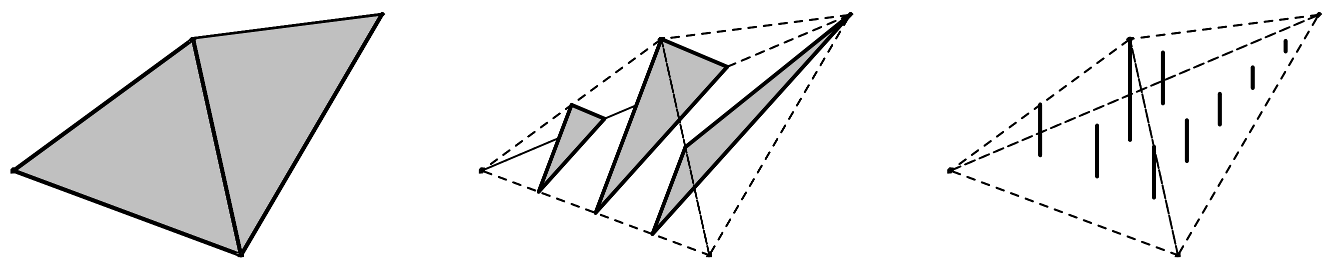
Sketch of the proof of Theorem 10

Recall the theorem: If P is k-integral, then

$$\mathrm{Vol}(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)) = \mathrm{SVol}^k(P).$$

i. We reduce the problem to the case of $k = 1$.

Idea: If P is k-integral (with $k > 1$), then each slice $\pi_{d-1}(\mathbf{y}, P)$ contributing to $\mathrm{SVol}^1(P)$ is $(k-1)$-integral.



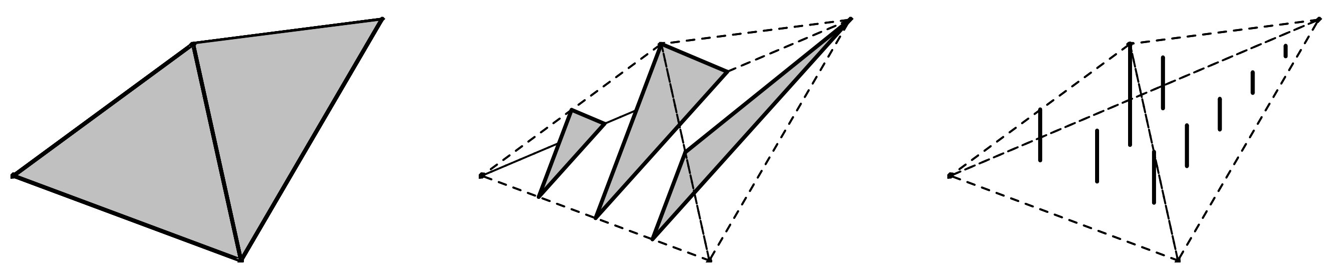
Sketch of the proof of Theorem 10

Recall the theorem: If P is k-integral, then

$$\mathrm{Vol}(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)) = \mathrm{SVol}^k(P).$$

i. We reduce the problem to the case of $k = 1$.

Idea: If P is k-integral (with $k > 1$), then each slice $\pi_{d-1}(\mathbf{y}, P)$ contributing to $\mathrm{SVol}^1(P)$ is $(k-1)$-integral.



ii. We reduce the problem to the case of a simplex with $k = 1$.

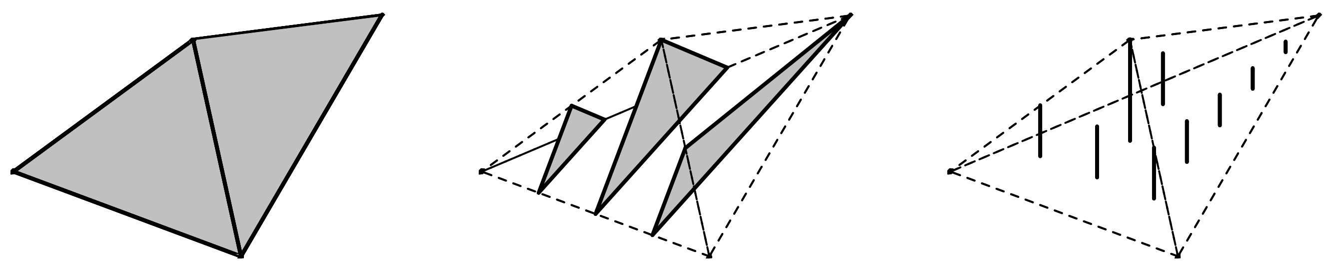
Sketch of the proof of Theorem 10

Recall the theorem: If P is k-integral, then

$$\mathrm{Vol}(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)) = \mathrm{SVol}^k(P).$$

i. We reduce the problem to the case of $k = 1$.

Idea: If P is k-integral (with $k > 1$), then each slice $\pi_{d-1}(\mathbf{y}, P)$ contributing to $\mathrm{SVol}^1(P)$ is $(k-1)$-integral.



ii. We reduce the problem to the case of a simplex with $k = 1$.

iii. Prove the case of a simplex with $k = 1$.

Possible Applications

- Prove positivity conjectures of special families of polytopes: Birkhoff polytopes, matroid polytopes.

Possible Applications

- Prove positivity conjectures of special families of polytopes: Birkhoff polytopes, matroid polytopes.
- Calculate the lower degree coefficients of Ehrhart polynomial quickly.

Possible Applications

- Prove positivity conjectures of special families of polytopes: Birkhoff polytopes, matroid polytopes.
- Calculate the lower degree coefficients of Ehrhart polynomial quickly.
- Algorithm to calculate volumes of rational polytopes: for any rational polytope P, one can always choose a coordinate system such that P is in 1-general position. Choose $D \in \mathbb{N}$ such that DP is integral. Then

$$\mathrm{Vol}(P) = \frac{1}{D} \sum_{\mathbf{y} \in \pi^{(d-1)}(P) \cap \frac{1}{D}\mathbb{Z}^1} \mathrm{Vol}_{d-1}(\pi_{d-1}(\mathbf{y}, P)).$$