Higher integrality conditions and volumes of slices

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Outline

• Basic definitions and theory of Ehrhart polynomials

• Motivation: Ehrhart polynomials of cyclic polytopes

• Main results
A (convex) polytope $P$ in the $d$-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \text{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n : \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$
Basic definitions

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Throughout this talk, we assume $P$ is full-dimensional, i.e., $\text{dim}(P) = d$. 
**Basic definitions**

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Throughout this talk, we assume $P$ is **full-dimensional**, i.e., $\dim(P) = d$.

The $d$-dimensional **lattice** $\mathbb{Z}^d \subset \mathbb{R}^d$ is the collection of all points with integer coordinates. Any point in the lattice is called a **lattice point** or an **integral point**.
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The $d$-dimensional \textit{lattice} $\mathbb{Z}^d \subset \mathbb{R}^d$ is the collection of all points with integer coordinates. Any point in the lattice is called a \textit{lattice point} or an \textit{integral point}.

A polytope $P$ is \textit{integral} if its vertices are all lattice points.
Higher integrality conditions and volumes of slices

Lattice points of a polytope
**Lattice points of a polytope**

**Definition 1.** For any polytope $P \subset \mathbb{R}^d$ and positive integer $m \in \mathbb{N}$, the *mth dilated polytope* of $P$ is $mP = \{mx : x \in P\}$. We denote by

$$i(P, m) = |mP \cap \mathbb{Z}^d|$$

the number of lattice points in $mP$. 
**Lattice points of a polytope**

**Definition 1.** For any polytope $P \subset \mathbb{R}^d$ and positive integer $m \in \mathbb{N}$, the *$m$th dilated polytope* of $P$ is $mP = \{mx : x \in P\}$. We denote by

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Example:
Examples of integral polytopes
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(i) When $d = 1$, $P$ is an interval $[a, b]$, where $a, b \in \mathbb{Z}$. Then $mP = [ma, mb]$ and

\[ i(P, m) = (b - a)m + 1. \]
Examples of integral polytopes

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$$i(P, m) = (b - a)m + 1.$$

(ii) When $d = 2$, $P$ is an integral polygon, and so is $mP$. Pick’s theorem states that for any integral polygon $Q$:

$$\text{area}(Q) = |Q \cap \mathbb{Z}^2| - \frac{1}{2}|\partial(Q) \cap \mathbb{Z}^2| - 1.$$

Hence,

$$i(P, m) = \text{area}(mP) + \frac{1}{2}|\partial(mP) \cap \mathbb{Z}^2| + 1 = \text{area}(P)m^2 + \frac{1}{2}|\partial(P) \cap \mathbb{Z}^2|m + 1$$
(iii) For any $d$, let $P$ be the convex hull of the set $\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1\}$, i.e. $P$ is the unit cube in $\mathbb{R}^d$. Then it is obvious that

$$i(P, m) = (m + 1)^d.$$
Theorem of Ehrhart (on integral polytopes)

**Theorem 2** (Ehrhart). Let $P$ be a $d$-dimensional integral polytope. Then $i(P, m)$ is a polynomial in $m$ of degree $d$.

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of $P$. 
Coefficients of Ehrhart polynomials

If $P$ is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, m)$?
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- The leading coefficient of $i(P, m)$ is the volume $\text{Vol}(P)$ of $P$.
- The second coefficient equals $1/2$ of the sum of the normalized volumes of each facet.
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- The constant term of $i(P, m)$ is always 1.
Coefficients of Ehrhart polynomials

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- No simple forms known for other coefficients for general polytopes.
Coefficients of Ehrhart polynomials

If $P$ is an integral polytope, what can we say about the coefficients of its Ehrhart polynomial $i(P, m)$?

- The leading coefficient of $i(P, m)$ is the volume $\text{Vol}(P)$ of $P$.
- The second coefficient equals $1/2$ of the sum of the normalized volumes of each facet.
- The constant term of $i(P, m)$ is always $1$.
- No simple forms known for other coefficients for general polytopes.
  - It is **NOT** even true that all the coefficients are nonnegative. For example, for the polytope $P$ with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(1, 1, 13)$, its Ehrhart polynomial is
    
    $$i(P, n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$
Questions

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➤ When can we have simple forms for all coefficients?
➤ When are they positive?
➤ When can the coefficients be described by volumes?
Beck, De Loera, Develin, Pfeifle and Stanley conjectured that the Ehrhart polynomial of an integral cyclic polytope has a simple formula.

Recall that given $n > d$, and $T = \{t_1 < \cdots < t_n\}$, a $d$-dimensional cyclic polytope $C_d(T) = C_d(t_1, \ldots, t_n)$ is the convex hull $\text{conv}\{\nu_d(t_1), \nu_d(t_2), \ldots, \nu_d(t_n)\}$ of the $n$ distinct points $\nu_d(t_i), 1 \leq i \leq n$, on the moment curve.

The moment curve (also known as rational normal curve) in $\mathbb{R}^d$ is defined by

$$\nu_d : \mathbb{R} \to \mathbb{R}^d, t \mapsto \nu_d(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix}.$$
Example: $T = \{1, 2, 3, 4\}$, $d = 3$:

$C_d(T)$ is the convex polytope whose vertices are

$$
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix},
\begin{pmatrix}
2 \\
4 \\
8
\end{pmatrix},
\begin{pmatrix}
3 \\
9 \\
27
\end{pmatrix},
\begin{pmatrix}
4 \\
16 \\
64
\end{pmatrix}.
$$
Theorem 3. Suppose $P = C_d(T)$ is a $d$-dimensional integral cyclic polytope. Then

$$i(P, m) = \sum_{i=0}^{d} \text{Vol}(\pi^{(d-i)}(P))m^i.$$ 

where $\pi^{(d-i)} : \mathbb{R}^d \rightarrow \mathbb{R}^i$ is the projection which drops the last $d - i$ coordinates.
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\begin{align*}
P = C_d(T) &= \text{conv}\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix} \} : \text{Vol}(P) = 2.
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\[ \pi^{(1)}(P) = \text{conv}\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \end{pmatrix} \} : \text{Vol}(\pi^{(1)}(P)) = 4. \]
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The theorem says:

\[
i(P, m) = 2m^3 + 4m^2 + 3m + 1.
\]
Question:

Are there other integral polytopes whose Ehrhart polynomials have the same form as cyclic polytopes? In other words, what kind of integral polytopes $P$ are there whose Ehrhart polynomials have the form

$$i(P, m) = \sum_{i=0}^{d} \text{Vol}(\pi^{(d-i)}(P))m^i?$$
Higher integrality conditions for affine spaces

**Definition 4.** An $\ell$-dimensional affine space $U \subset \mathbb{R}^d$ is *integral* if

$$\pi^{(d-\ell)}(U \cap \mathbb{Z}^d) = \mathbb{Z}^\ell.$$ 

Or equivalently, the projection $\pi^{(d-\ell)}$ induces a bijection between $U \cap \mathbb{Z}^d$ and $\mathbb{Z}^\ell$. 
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**Example:**

(i) \(\ell = 0\) : \(U\) is integral if and only if \(U\) is a lattice point.
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**Example:**

(i) $\ell = 0 : U$ is integral if and only if $U$ is a lattice point.

(ii) $\ell = 1 : \text{In } \mathbb{R}^2$,
**Higher integrality conditions for affine spaces**

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**Example:**

(i) \( \ell = 0 \): \( U \) is integral if and only if \( U \) is a lattice point.

(ii) \( \ell = 1 \): In \( \mathbb{R}^2 \),

In general, \( U \) is integral if and only if \( U \) contains a lattice point and \( \text{dir}(U) = (1, z_2, \ldots, z_d) \in \mathbb{Z}^d \).
A property of integral cyclic polytopes

For any integral cyclic polytope $P$, we have that

\[\text{any affine space determined by a subset of } \text{Vert}(P) \text{ is integral}.\]  

(★)
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**Theorem 5.** Suppose $P$ is a polytope satisfying (\star). Then

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i(P, m) = \sum_{i=0}^{d} \text{Vol}\left(\pi^{(d-i)}(P)\right)m^i.
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**Question:**

(i) Can we relax \((\star)\) to the following condition?

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\text{Any affine space } \text{aff}(F) \text{ determined by a face } F \text{ of } P \text{ is integral.} \quad (\triangle)
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A property of integral cyclic polytopes

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any affine space determined by a subset of $\text{Vert}(P)$ is integral.  \hfill (\star)

**Theorem 5.** Suppose $P$ is a polytope satisfying $(\star)$. Then

$$i(P, m) = \sum_{i=0}^{d} \text{Vol}(\pi^{(d-i)}(P))m^i.$$  

Question:

(i) Can we relax $(\star)$ to the following condition?

Any affine space $\text{aff}(F)$ determined by a face $F$ of $P$ is integral.  \hfill $(\triangle)$

(ii) What if $(\star)$ or $(\triangle)$ is only satisfied for affine spaces of dimension in a subset $S \subseteq \{0, 1, \ldots, d\}$?
Higher integrality conditions for polytopes

Definition 6. A polytope $P$ is $k$-integral if for any $0 \leq \ell \leq k$, we have that $\text{aff}(F)$ is integral for any $\ell$-dimensional face $F$ of $P$.

We say $P$ is fully integral if $k = d$. 

Higher integrality conditions for polytopes

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We say $P$ is **fully integral** if $k = d$.

**Example:**
(i) $k = 0$ : $P$ is 0-integral if and only if $P$ is integral.
Higher integrality conditions for polytopes

Definition 6. A polytope \( P \) is \( k \)-integral if for any \( 0 \leq \ell \leq k \), we have that \( \text{aff}(F) \) is integral for any \( \ell \)-dimensional face \( F \) of \( P \).

We say \( P \) is fully integral if \( k = d \).

Example:

(i) \( k = 0 \) : \( P \) is 0-integral if and only if \( P \) is integral.

(ii) \( k = 1 \) : \( P \) is 1-integral if and only if \( P \) is integral and \( \text{dir}(e) = (1, z_2, \ldots, z_d) \in \mathbb{Z}^d \) for any edge \( e \) of \( P \).
Higher integrality conditions for polytopes

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Conjecture 7. If $P$ is $k$-integral, then for $0 \leq \ell \leq k$, the coefficient of $m^\ell$ in $i(P, m)$ is $\text{Vol}(\pi^{d-\ell}(P))$. 
Higher integrality conditions for polytopes

**Definition 6.** A polytope $P$ is *$k$-integral* if for any $0 \leq \ell \leq k$, we have that $\text{aff}(F)$ is integral for any $\ell$-dimensional face $F$ of $P$.

We say $P$ is *fully integral* if $k = d$.

**Example:**

(i) $k = 0 : P$ is 0-integral if and only if $P$ is integral.

(ii) $k = 1 : P$ is 1-integral if and only if $P$ is integral and $\text{dir}(e) = (1, z_2, \ldots, z_d) \in \mathbb{Z}^d$ for any edge $e$ of $P$.

**Conjecture 7.** If $P$ is $k$-integral, then for $0 \leq \ell \leq k$, the coefficient of $m^\ell$ in $i(P, m)$ is $\text{Vol}(\pi^{d-\ell}(P))$.

**Example:** $P = \text{conv}\{(0, 0, 0), (4, 0, 0), (3, 6, 0), (2, 2, 2)\}$. One can check that $P$ is 1-integral.

$$i(P, m) = 8m^3 + 10m^2 + 4m + 1,$$

and $\text{Vol}(\pi^{d-1}(P)) = \text{Vol}([0, 4]) = 4$ and $\text{Vol}(\pi^{(d-0)}(P)) = \text{Vol}(\mathbb{R}^0) = 1$. 
Definition 8. For any $y \in \pi^{(d-k)}(P)$, we define *the slice of $P$ over $y$*, denoted by $\pi_{d-k}(y, P)$, to be the intersection of $P$ with the inverse image of $y$ under $\pi^{(d-k)}$. 
Slices of a polytope

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Example: $P = \text{conv}\{(0, 0, 0), (4, 0, 0), (3, 6, 0), (2, 2, 2)\}$.
Recall $\pi^{(d-1)}(P) = [0, 4]$ and $i(P, m) = 8m^3 + 10m^2 + 4m + 1$. 
**Definition 8.** For any \( y \in \pi^{(d-k)}(P) \), we define the slice of \( P \) over \( y \), denoted by \( \pi_{d-k}(y, P) \), to be the intersection of \( P \) with the inverse image of \( y \) under \( \pi^{(d-k)} \).

**Example:** \( P = \text{conv}\{(0, 0, 0), (4, 0, 0), (3, 6, 0), (2, 2, 2)\} \).

Recall \( \pi^{(d-1)}(P) = [0, 4] \) and \( i(P, m) = 8m^3 + 10m^2 + 4m + 1 \).

\[
\begin{align*}
i(\pi_2(0, P), m) &= 1, \\
i(\pi_2(1, P), m) &= m^2 + 2m + 1, \\
i(\pi_2(2, P), m) &= 4m^2 + 4m + 1, \\
i(\pi_2(3, P), m) &= 3m^2 + 4m + 1 \quad \text{and} \\
i(\pi_2(4, P), m) &= 1. \\
\end{align*}
\]

Their sum is \( 8m^2 + 10m + 5 \).
Main theorems

Theorem 9. If $P$ is $k$-integral, then the coefficient of $m^\ell$ in $i(P, m)$ is

$$
\begin{cases}
\text{Vol}(\pi^{d-\ell}(P)) & \text{if } 0 \leq \ell \leq k, \\
\sum_{y \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{coefficient of } m^{\ell-k} \text{ in } i(\pi_{d-k}(y, P), m) & \text{if } k + 1 \leq \ell \leq d
\end{cases}
$$
Main theorems

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\end{cases}
$$

**Theorem 10.** Suppose $k < d$. If $P$ is $k$-integral, then

$$
\text{Vol}(P) = \sum_{y \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{Vol}_{d-k}(\pi_{d-k}(y, P))
$$

where $\text{Vol}_{d-k}$ is the volume with respect to the lattice $\mathbb{Z}^{d-k}$. 
Main theorems

Theorem 9. If \( P \) is \( k \)-integral, then the coefficient of \( m^{\ell} \) in \( i(P, m) \) is

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\begin{cases}
\text{Vol}(\pi^{d-\ell}(P)) & \text{if } 0 \leq \ell \leq k, \\
\sum_{y \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{coefficient of } m^{\ell-k} \text{ in } i(\pi_{d-k}(y, P), m) & \text{if } k + 1 \leq \ell \leq d
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\text{Vol}(P) = \sum_{y \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{Vol}_{d-k}(\pi_{d-k}(y, P)),
\]

where \( \text{Vol}_{d-k} \) is the volume with respect to the lattice \( \mathbb{Z}^{d-k} \).

Definition 11. We define the \( k \)-th \( S \)-volume of \( P \) to be

\[
\text{SVol}_k(P) = \sum_{y \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{Vol}_{d-k}(\pi_{d-k}(y, P)).
\]
Remark 12. $SVol^0(P) = Vol(P)$ and $SVol^d(P) = |P \cap Z^d|$. 
**Remarks**

**Remark 12.** $S\text{Vol}^0(P) = \text{Vol}(P)$ and $S\text{Vol}^d(P) = |P \cap Z^d|$.

**Remark 13.** Theorem 10 says if $k < d$ and $P$ is $k$-integral, then $\text{Vol}(P) = S\text{Vol}^k(P)$. Note that $P$ is $\ell$-integral for any $\ell \leq k$, so we have

$$\text{Vol}(P) = S\text{Vol}^0(P) = S\text{Vol}^1(P) = \cdots = S\text{Vol}^k(P).$$
Remark 14. The condition $k$-integral in Theorem 10 can be relaxed to $(k - 1)$-integral and in $k$-general position.
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Example: 1-dimensional affine space in $\mathbb{R}^2$. integral vs in general position:

In general, $U$ is in general position if and only if $\text{dir}(U) = (1, y_1, \ldots, y_d) \in \mathbb{R}^d$. 
Remark 14. The condition $k$-integral in Theorem 10 can be relaxed to $(k - 1)$-integral and \textit{in $k$-general position}.

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Theorem 9 can be reduced to Theorem 10.
Higher integrality conditions and volumes of slices

Fu Liu

**Reduction to volume formula**

i. If $P$ is fully integral, then in particular $P$ is $(d - 1)$-integral. For any $y \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}$, the slice $\pi_1(y, P)$ is either a 1-dimensional integral polytope, or a lattice point. In either case, we have that $|\pi_1(y, P) \cap \mathbb{Z}^d| = 1 + \text{Vol}_1(\pi_1(y, P))$. 
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Then

$$|P \cap \mathbb{Z}^d| = \sum_{y \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}} |\pi_1(y, P) \cap \mathbb{Z}^d|$$

$$= \sum_{y \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}} (1 + \text{Vol}_1(\pi_1(y, P))) = |\pi^{(1)}(P) \cap \mathbb{Z}^{d-1}| + \text{Vol}(P).$$
However, $\pi^{(1)}(P)$ is fully integral. Hence,

$$|P \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \text{Vol} (\pi^{(d-i)}(P)).$$

Note that $P$ is $k$-integral $\Rightarrow mP$ is $k$-integral as well. Therefore,

$$i(P, m) = |mP \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \text{Vol} (\pi^{(d-i)}(mP)) = \sum_{i=0}^{d} \text{Vol} (\pi^{(d-i)}(P))m^i.$$
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ii. If $k \leq d-1$, the projection $\pi^{(d-k)}(P)$ is fully integral and for any $y \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k$, the slice $\pi_{d-k}(y, P)$ is an integral polytope.
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We prove Theorem 9 by using the result on fully integral polytopes as well as a local formula relating the number of lattice points to volumes of faces for integral polytopes obtained by Morelli, McMullen, Pommersheim-Thomas, Berline-Vergne.
Sketch of the proof of Theorem 10

Recall the theorem: If $P$ is $k$-integral, then

$$
\text{Vol}(P) = \sum_{y \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{Vol}_{d-k}(\pi_{d-k}(y, P)) = S\text{Vol}^k(P).
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i. We reduce the problem to the case of $k = 1$.

Idea: If $P$ is $k$-integral (with $k > 1$), then each slice $\pi_{d-1}(y, P)$ contributing to $S\text{Vol}^1(P)$ is $(k - 1)$-integral.
Sketch of the proof of Theorem 10

Recall the theorem: If $P$ is $k$-integral, then

$$\text{Vol}(P) = \sum_{y \in \pi_{d-k}(P) \cap \mathbb{Z}^k} \text{Vol}_{d-k}(\pi_{d-k}(y, P)) = \text{SVol}^k(P).$$

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**Idea:** If $P$ is $k$-integral (with $k > 1$), then each slice $\pi_{d-1}(y, P)$ contributing to $\text{SVol}^1(P)$ is $(k - 1)$-integral.

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iii. Prove the case of a simplex with $k = 1$. 
Possible Applications

Prove positivity conjectures of special families of polytopes: Birkhoff polytopes, matroid polytopes.
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- Calculate the lower degree coefficients of Ehrhart polynomial quickly.
- Algorithm to calculate volumes of rational polytopes: for any rational polytope $P$, one can always choose a coordinate system such that $P$ is in 1-general position.

Choose $D \in \mathbb{N}$ such that $DP$ is integral. Then

$$\text{Vol}(P) = \frac{1}{D} \sum_{y \in \pi^{(d-1)}(P) \cap \frac{1}{D}\mathbb{Z}^1} \text{Vol}_{d-1}(\pi_{d-1}(y, P)).$$