The lecture hall parallelopiped

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This is joint work with Richard Stanley.

PART I:

Definitions and Backgrounds

***Summary*:** We will introduce δ-vectors, Eulerian numbers, $\mathbf{s}$-lecture-hall polytopes and parallelepiped, and discuss our goal.

Ehrhart polynomials and δ-vectors

Suppose that P is an n-dimensional integral polytope, i.e., a (convex) polytope whose vertices have integer coordinates. Let

$$i(P,t) := |tP \cap \mathbb{Z}^D|$$

be the number of lattice points in the tth dilation tP of P.

Then $i(P,t)$ is a polynomial in t of degree n, called the *Ehrhart polynomial* of P.

Ehrhart polynomials and δ-vectors

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It is well-known that the generating function of $i(P, t)$ has the form

$$\sum_{t \geq 0} i(P, t) z^t = \frac{\delta_P(z)}{(1 - z)^{n+1}},$$

where $\delta_P(z)$ is a polynomial of degree at most n with nonnegative integer coefficients.

We denote by $\delta_{P,i}$ the coefficient of z^i in $\delta_P(z)$, for $0 \leq i \leq n$.

We call $(\delta_{P,0}, \delta_{P,1}, \ldots, \delta_{P,n})$ the *δ-vector* or *h^*-vector* of P.

Descents, inversion sequences and Eulerian numbers

Let $\boldsymbol{r} = (r_1, \dots, r_n)$ be a sequence of nonnegative integers. We say that i is a *(regular) descent* of $\boldsymbol{r}$ if $r_i > r_{i+1}$. Define the *descent set* $\mathrm{Des}(\boldsymbol{r})$ of $\boldsymbol{r}$ by

$$\mathrm{Des}(\boldsymbol{r}) = \{i \mid r_i > r_{i+1}\},$$

and define its size $\mathrm{des}(\boldsymbol{r}) = \#\,\mathrm{Des}(\boldsymbol{r})$.

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For any nonnegative integer n, let $\langle n \rangle := \{0, 1, \dots, n\}$.

An *inversion sequence of length n* is any element in the set

$$\langle n-1 \rangle \times \cdots \times \langle 1 \rangle \times \langle 0 \rangle.$$

We refer to the above set as *the set of inversion sequences of length n*.

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The *Eulerian number* $A(n, i)$ is the number of inversion sequences of length n with $i - 1$ descents.

An example

Example. There are $3! = 6$ inversion sequences of length 3 :

$\boldsymbol{r}$	$\mathrm{Des}(\boldsymbol{r})$	$\mathrm{des}(\boldsymbol{r})$
$(0,0,0)$	$\varnothing$	$0;$
$(0,1,0)$	$\{2\}$	$1;$
$(1,0,0)$	$\{1\}$	$1;$
$(1,1,0)$	$\{2\}$	$1;$
$(2,0,0)$	$\{1\}$	$1;$
$(2,1,0)$	$\{1,2\}$	$2.$

Hence, $A(3,1) = 1$, $A(3,2) = 4$, $A(3,3) = 1$, and $A(3,i) = 0$ for $i \geq 4$.

The δ-vector of a unit cube

The δ-vector of the n-dimensional unit cube, denoted by $\square_n$, is given by

$$\delta_{\square_n, i} = A(n, i+1) = \#\text{inversion sequences of length } n \text{ with } i \text{ descents}.$$

Example.

$$\delta_{\square_3} = (1, 4, 1, 0).$$

Hence,

$$\sum_{t \geq 0} i(\square_3, t)\, z^t = \frac{1 + 4z + z^2}{(1-z)^4}.$$

The $\mathbf{s}$-lecture hall polytope

Let $\mathbf{s} = (s_1, \ldots, s_n)$ be a sequence of positive integers. The $\mathbf{s}$-*lecture hall polytope*, denoted by $P_{\mathbf{s}}$, is the polytope in $\mathbb{R}^n$ defined by the inequalities

$$0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \cdots \leq \frac{x_n}{s_n} \leq 1.$$

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Corteel-Lee-Savage showed that $P_{(1,2,\ldots,n)}$ and $P_{(n,n-1,\ldots,1)}$ have the same Ehrhart polynomials as the n-dimensional unit cube $\square_n$.

Hence,

$$\delta_{P_{(1,2,\ldots,n)},i} = \delta_{P_{(n,n-1,\ldots,1)},i} = \delta_{\square_n,i} = A(n, i+1)$$

$=$ $\#$inversion sequences of length n with i descents .

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Goal:

Give a bijective proof for the above fact.

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Hence,

$$\begin{aligned} \delta_{P_{(1,2,\ldots,n)},i} = \delta_{P_{(n,n-1,\ldots,1)},i} &= \delta_{\square_n,i} = A(n, i+1) \\ &= \#\text{inversion sequences of length } n \text{ with } i \text{ descents}. \end{aligned}$$

Goal:

Give a bijective proof for the above fact.

In this talk, we will only discuss the bijection for $\mathbf{s} = (n, n-1, \ldots, 1)$.

The $\mathbf{s}$-lecture hall polytope (cont'd)

It is easy to see that $P_{\mathbf{s}}$ has the vertex set

$$\{(0,0,0,\dots,0),(0,0,\dots,0,s_n),(0,0,\dots,0,s_{n-1},s_n),\dots,(s_1,s_2,\dots,s_n)\}.$$

Hence $P_{\mathbf{s}}$ is a simplex.

The δ-vector of a simplex P can be described in terms of number of lattice points in a fundamental parallelepiped associated to P.

δ-vector of simplices

For a set of independent vectors $W = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$, we define the *fundamental (half-open) parallelepiped generated by W* to be

$$\mathrm{Par}(W) = \mathrm{Par}(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n) := \left\{ \sum_{i=1}^{n} c_i \boldsymbol{w}_i \mid 0 \le c_i < 1 \right\}.$$

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For any set $S \subset \mathbb{R}^N$, we denote by $\mathcal{L}^i(S)$ the set of lattice points in S whose last coordinates are i :

$$\mathcal{L}^i(S) := \left\{ \mathbf{x} \in S \cap \mathbb{Z}^N \mid \text{last coordinate of } \mathbf{x} \text{ is } i \right\},$$

and let $\ell^i(S) := \#\mathcal{L}^i(S)$ be the cardinality of $\mathcal{L}^i(S)$.

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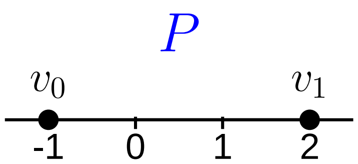
For convenience, for any vector $\boldsymbol{v} \in \mathbb{R}^N$, we let $\boldsymbol{v}^* := (\boldsymbol{v}, 1)$ be the vector obtained by appending 1 to the end of $\boldsymbol{v}$.

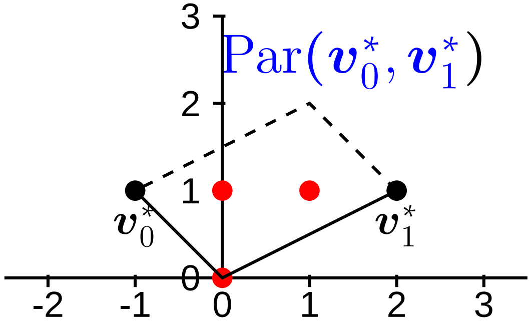
δ-vector of simplices (cont'd)

Suppose P is an n-dimensional simplex with vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Then the δ-vector of P is given by

$$\delta_{P,i} = \ell^i(\mathrm{Par}(\boldsymbol{v}_0^*, \dots, \boldsymbol{v}_n^*)), \quad 0 \le i \le n.$$

Example.





$$i(P,t) = 3t+1, \qquad \sum_{t \ge 0} i(P,t) z^t = \frac{2t+1}{(1-t)^2}.$$

$\mathbf{s}$-lecture hall parallelepiped

Definition 1. Given a sequence $\mathbf{s} = (s_1, \dots, s_n)$ of positive integers, the $\mathbf{s}$-*lecture hall parallelepiped*, denoted by $\mathrm{Par}_{\mathbf{s}}$, is the fundamental parallelepiped generated by the non-origin vertices of the $\mathbf{s}$-lecture hall polytope $P_{\mathbf{s}}$:

$$\mathrm{Par}_{\mathbf{s}} := \mathrm{Par}((0, 0, \dots, 0, s_n), (0, 0, \dots, 0, s_{n-1}, s_n), \dots, (s_1, s_2, \dots, s_n)).$$

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Lemma 2. *The δ-vector of $P_{\mathbf{s}}$ is given by*

$$\delta_{P_{\mathbf{s}}, i} = \ell^i(\mathrm{Par}_{\mathbf{s}^*}), \quad 0 \le i \le n.$$

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Furthermore, if $s_n = 1$, then

$$\ell^i(\mathrm{Par}_{\mathbf{s}}) = \ell^i(\mathrm{Par}_{\mathbf{s}^*}), \quad 0 \le i \le n.$$

Hence,

$$\delta_{P_{\mathbf{s}},i} = \ell^i(\mathrm{Par}_{\mathbf{s}}), \quad 0 \le i \le n.$$

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We will present results on the cases where $s_n = 1$, where $\mathbf{s} = (n, n-1, \dots, 1)$ is a special case.

Detailed Goal

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When $\mathbf{s} = (n, n-1, \ldots, 1)$, can we give a nice bijection between $\mathcal{L}^i(\mathrm{Par}_{\mathbf{s}})$ and inversion sequences of length n with i descents?

In fact, we will construct bijections between $\mathcal{L}^i(\mathrm{Par}_{\mathbf{s}})$ and certain sequences with i $\mathbf{s}$-*descents*, for any $\mathbf{s}$ with $s_n = 1$.

PART II:

The Bijection

***Summary*:** We will construct a bijection from lattice points in $\mathrm{Par}_{\mathbf{s}}$ to certain family of sequences, and show it has the desired property.

The map $\mathrm{REM}_{\mathbf{s}}$

Assume that $\mathbf{s} = (s_1, \dots, s_n)$ is a sequence of positive integers. We associate the following set to $\mathbf{s}$:

$$\Psi_{\mathbf{s}} = \langle s_1 - 1 \rangle \times \cdots \times \langle s_n - 1 \rangle.$$

Notice that $\Psi_{\mathbf{s}}$ is the set of inversion sequences of length n if $\mathbf{s} = (n, n-1, \dots, 1)$.

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Definition 3. We define a map

$$\mathrm{REM}_{\mathbf{s}} : \mathrm{Par}_{\mathbf{s}} \cap \mathbb{Z}^n \to \Psi_{\mathbf{s}}$$

in the following way. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathrm{Par}_{\mathbf{s}} \cap \mathbb{Z}^n$. For each x_i, let $k_i = \left\lfloor \frac{x_i}{s_i} \right\rfloor$ be the quotient of dividing x_i by s_i, and r_i be the remainder. Hence

$$x_i = k_i s_i + r_i,$$

where $k_i \in \langle n-1\rangle$ and $r_i \in \langle s_i - 1\rangle$. Let $\mathbf{k} = (k_1, \ldots, k_n)$ be the *quotient sequence* and $\boldsymbol{r} = (r_1, \ldots, r_n)$ be the *remainder sequence*. Then we define $\mathrm{REM}_{\mathbf{s}}(\mathbf{x}) = \boldsymbol{r}$.

An example of the map $\mathrm{REM}_{\mathbf{s}}$

Example. Let $\mathbf{s} = (3,2,1)$. We have

$\mathbf{x} \in \mathrm{Par}_{\mathbf{s}} \cap \mathbb{Z}^n$	$\mathbf{k}$	$\boldsymbol{r} = \mathrm{REM}_{\mathbf{s}}(\mathbf{x})$
$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
$(0,1,1)$	$(0,0,1)$	$(0,1,0)$
$(1,2,1)$	$(0,1,1)$	$(1,0,0)$
$(1,1,1)$	$(0,0,1)$	$(1,1,0)$
$(2,2,1)$	$(0,1,1)$	$(2,0,0)$
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Note that the last column consists of each element of $\Psi_{\mathbf{s}} = \langle 2 \rangle \times \langle 1 \rangle \times \langle 0 \rangle$ exactly once.

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Lemma 4. $\mathrm{REM}_{\mathbf{s}}$ *is a bijection from* $\mathrm{Par}_{\mathbf{s}} \cap \mathbb{Z}^n$ *to* $\Psi_{\mathbf{s}}$*.*

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Question How do we recover the quotient sequence $\mathbf{k}$?

$\mathbf{s}$-descents

Definition 5. Let $\boldsymbol{r} = (r_1, \dots, r_n)$. We say that i is an $\mathbf{s}$-*descent of* $\boldsymbol{r}$ if $\dfrac{r_i}{s_i} > \dfrac{r_{i+1}}{s_{i+1}}$.

We denote by $\mathrm{Des}_{\mathbf{s}}(\boldsymbol{r})$ the set of $\mathbf{s}$-descents of $\boldsymbol{r}$, and let $\mathrm{des}_{\mathbf{s}}(\boldsymbol{r}) = \# \mathrm{Des}_{\mathbf{s}}(\boldsymbol{r})$ be its cardinality. For any $1 \le i \le n$, we let

$$\mathrm{des}_{\mathbf{s}}^{<i}(\boldsymbol{r}) = \#\mathbf{s}\text{-descents of } \boldsymbol{r} \text{ that are smaller than } i$$

s-descents

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Example. Let $\boldsymbol{r} = (1, 1, 0)$.

For $\mathrm{s} = (3, 2, 1)$: since $\frac{1}{3} < \frac{1}{2} > \frac{0}{1}$, we have

$$\mathrm{Des}_{\mathbf{s}}(\boldsymbol{r}) = \{2\}, \text{ and } \mathrm{des}_{\mathbf{s}}^{<1}(\boldsymbol{r}) = 0, \mathrm{des}_{\mathbf{s}}^{<2}(\boldsymbol{r}) = 0, \mathrm{des}_{\mathbf{s}}^{<3}(\boldsymbol{r}) = 1.$$

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Note that $\mathrm{Des}(\boldsymbol{r}) = \{2\}$.

Lemma 6. *If* $\mathbf{s} = (n, n-1, \ldots, 1)$, *then* $\mathbf{s}$-*descents of* $\boldsymbol{r} \in \Psi_{\mathbf{s}}$ *are the same as regular descents of* $\boldsymbol{r}$.

An example of $\mathbf{s}$-descents

Example. Let $\mathbf{s} = (3, 2, 1)$. We have

$\boldsymbol{r} \in \Psi_{\mathbf{s}}$	$\mathrm{Des}_{\mathbf{s}}(\boldsymbol{r}) = \mathrm{Des}(\boldsymbol{r})$	$(\mathrm{des}_{\mathbf{s}}^{<}1(\boldsymbol{r}), \mathrm{des}_{\mathbf{s}}^{<2}(\boldsymbol{r}), \mathrm{des}_{\mathbf{s}}^{<3}(\boldsymbol{r}))$
$(0,0,0)$	$\emptyset$	$(0,0,0);$
$(0,1,0)$	$\{2\}$	$(0,0,1);$
$(1,0,0)$	$\{1\}$	$(0,1,1);$
$(1,1,0)$	$\{2\}$	$(0,0,1);$
$(2,0,0)$	$\{1\}$	$(0,1,1);$
$(2,1,0)$	$\{1,2\}$	$(0,1,2).$

An example of $\mathbf{s}$-descents

Example. Let $\mathbf{s} = (3, 2, 1)$. We have

$\boldsymbol{r} \in \Psi_{\mathbf{s}}$	$\mathrm{Des}_{\mathbf{s}}(\boldsymbol{r}) = \mathrm{Des}(\boldsymbol{r})$	$(\mathrm{des}_{\mathbf{s}}^{<} 1(\boldsymbol{r}), \mathrm{des}_{\mathbf{s}}^{<2}(\boldsymbol{r}), \mathrm{des}_{\mathbf{s}}^{<3}(\boldsymbol{r}))$
$(0,0,0)$	$\varnothing$	$(0,0,0)$;
$(0,1,0)$	$\{2\}$	$(0,0,1)$;
$(1,0,0)$	$\{1\}$	$(0,1,1)$;
$(1,1,0)$	$\{2\}$	$(0,0,1)$;
$(2,0,0)$	$\{1\}$	$(0,1,1)$;
$(2,1,0)$	$\{1,2\}$	$(0,1,2)$.

Lemma 7. *Suppose* $\boldsymbol{r} = \mathrm{REM}_{\mathbf{s}}(\mathbf{x})$ *is the remainder sequence of dividing* $\mathbf{x}$ *by* $\mathbf{s}$. *Then the quotient sequence is given by*

$$\mathbf{k} = (\mathrm{des}_{\mathbf{s}}^{<} 1(\boldsymbol{r}), \mathrm{des}_{\mathbf{s}}^{<2}(\boldsymbol{r}), \ldots, \mathrm{des}_{\mathbf{s}}^{<n}(\boldsymbol{r})).$$

Inverse of $\mathrm{REM}_{\mathbf{s}}$

Since $x_i = k_i s_i + r_i$, we construct the inverse of $\mathrm{REM}_{\mathbf{s}}$.

Theorem 8. *The inverse of the map* $\mathrm{REM}_{\mathbf{s}}$ *is:*

$$\begin{aligned} \mathrm{REM}_{\mathbf{s}}^{-1} : \Psi_{\mathbf{s}} &\to \mathrm{Par}_{\mathbf{s}} \cap \mathbb{Z}^n \\ \boldsymbol{r} = (r_1, \ldots, r_n) &\mapsto (\mathrm{des}_{\mathbf{s}}^{<1}(\boldsymbol{r}) s_1 + r_1, \ldots, \mathrm{des}_{\mathbf{s}}^{<n}(\boldsymbol{r}) s_n + r_n) \end{aligned}$$

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Note that $\mathrm{des}_{\mathbf{s}}^{<n}(\boldsymbol{r}) = \mathrm{des}_{\mathbf{s}}(\boldsymbol{r})$. When $s_n = 1$, we have $r_n = 0$, and thus the last entry in $\mathrm{REM}_{\mathbf{s}}^{-1}(\boldsymbol{r})$ is $\mathrm{des}_{\mathbf{s}}(\boldsymbol{r})$.

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Corollary 9. *If $s_n = 1$, the map $\mathrm{REM}_{\mathbf{s}}$ induces a bijection from $\mathcal{L}^i(\mathrm{Par}_{\mathbf{s}})$ to the elements $\boldsymbol{r} \in \Psi_{\mathbf{s}}$ with exactly i s-descents. Hence,*

$$\ell^i(\mathrm{Par}_{\mathbf{s}}) = \#\{\boldsymbol{r} \in \Psi_{\mathbf{s}} \mid \mathrm{des}_{\mathbf{s}}(\boldsymbol{r}) = i\}.$$

In particular, if $\mathbf{s} = (n, n-1, \dots, 1)$, the map $\mathrm{REM}_{\mathbf{s}}$ gives a bijection between $\mathcal{L}^i(\mathrm{Par}_{\mathbf{s}})$ and inversion sequences of length n with i descents.

Results on δ-vectors

Theorem 10. *Suppose that* $\mathrm{s} = (s_1, \ldots, s_n)$ *is a sequence of positive integers with* $s_n = 1$*. Then the* δ*-vector of the* s*-lecture hall polytope* P_{s} *is given by*

$$\delta_{P_{\mathbf{s}},i} = \#\{\boldsymbol{r} \in \Psi_{\mathbf{s}} \mid \mathrm{des}_{\mathbf{s}}(\boldsymbol{r}) = i\}, \quad 0 \le i \le n.$$

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$$\delta_{P_{\mathbf{s}},i} = \#\{\boldsymbol{r} \in \Psi_{\mathbf{s}} \mid \operatorname{des}_{\mathbf{s}}(\boldsymbol{r}) = i\}, \quad 0 \le i \le n.$$

Theorem 11. *Suppose that* $\mathbf{s} = (s_1, \dots, s_n)$ *is a sequence of positive integers. Then the* δ*-vector of the* $\mathbf{s}$*-lecture hall polytope* $P_{\mathbf{s}}$ *is given by*

$$\delta_{P_{\mathbf{s}},i} = \#\{\boldsymbol{r} \in \Psi_{\mathbf{s}} \times \langle 0 \rangle \mid \operatorname{des}_{\mathbf{s}^*}(\boldsymbol{r}) = i\}, \quad 0 \le i \le n.$$