A combinatorial analysis of Severi degrees

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Outline

- Background on Severi degrees (classical and generalized ones)
- Computing Severi degrees via long-edge graphs
  - Introduce combinatorial objects in Fomin-Mikhalkin's formula for computing classical Severi degrees
  - Two main results: Vanishing Lemma and Linearity Theorem
  - First application
- Severi degrees on toric surfaces (joint work with Brian Osserman)
  - Introduce Ardila-Block's formula for computing Severi degrees for certain toric surfaces
  - Second application

PART I:

Background on Severi degrees

***Summary*:** We introduce classical and generalized Severi degrees and relevant results, finishing with the original motivation of this work.

Classical Severi degree

- $N^{d,\delta}$ counts the number of curves of degree d with δ nodes passing through $\frac{d(d+3)}{2} - \delta$ general points in $\mathbb{CP}^2$.
- $N^{d,\delta}$ is the degree of the Severi variety.
- $N^{d,\delta} = N_{d,\frac{(d-1)(d-2)}{2}-\delta}$ (Gromov-Witten invariant) when $d \geq \delta + 2$.

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Generalized Severi degree

Let $\mathscr{L}$ be a line bundle on a complex projective smooth surface Y.

- $N^{\delta}(Y, \mathscr{L})$ counts the number of δ-nodal curves in $\mathscr{L}$ passing through $\dim |\mathscr{L}| - \delta$ points in general position.
- $N^{\delta}(\mathbb{CP}^2, \mathscr{O}_{\mathbb{CP}^2}(d)) = N^{d,\delta}$.

Polynomiality of $N^{d,\delta}$

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- In 2009, Fomin and Mikhalkin showed that $N^{d,\delta}$ is given by a *node polynomial* $N_\delta(d)$ for $d \geq 2\delta$.

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- In 2011, Block improved the threshold bound to $d \geq \delta$.
- In 2012, Kleiman and Shende lowered the bound further to $d \geq \lceil \delta/2 \rceil + 1$.

Göttsche's conjecture

In 1998, Göttsche conjectured the following:

(i) For every fixed δ, there exists a **universal polynomial** $T_\delta(w, x, y, z)$ of degree δ such that

$$N^\delta(Y, \mathscr{L}) = T_\delta(\mathscr{L}^2, \mathscr{L} \cdot \mathscr{K}, \mathscr{K}^2, c_2)$$

whenever Y is smooth and $\mathscr{L}$ is $(5\delta - 1)$-ample, where $\mathscr{K}$ and c_2 are the canonical class and second Chern class of Y, respectively.

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(ii) Moreover, there exist power series $B_1(q)$ and $B_2(q)$ such that

$$\sum_{\delta \geq 0} T_\delta(x, y, z, w)(DG_2(q))^\delta = \frac{(DG_2(q)/q)^{\frac{z+w}{12} + \frac{x-y}{2}} B_1(q)^z B_2(q)^y}{(\Delta(q) D^2 G_2(q)/q^2)^{\frac{z+w}{24}}},$$

where $G_2(q) = -\frac{1}{24} + \sum_{n>0} \left(\sum_{d|n} d \right) q^n$ is the second Eisenstein series, $D = q\frac{d}{dq}$ and $\Delta(q) = q \prod_{k>0}(1 - q^k)^{24}$ is the modular discriminant. The above formula is known as the *Göttsche-Yau-Zaslow formula*.

Göttsche's conjecture (cont'd)

- In 2010, Tzeng proved Göttsche's conjecture (both parts).
- In 2011, Kool, Shende and Thomas proved part (i) of Göttsche's conjecture, i.e., the assertion of the existence of a universal polynomial, with a sharper bound on the necessary threshold on the ampleness of $\mathscr{L}$.

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Connection to node polynomial

$N^{d,\delta} = N^{\delta}(Y, \mathscr{L})$ when $Y = \mathbb{CP}^2, \mathscr{L} = \mathscr{O}_{\mathbb{CP}^2}(d)$, in which case the four topological numbers become:

$$\mathscr{L}^2 = d^2, \mathscr{L} \cdot \mathscr{K} = -3d, \mathscr{K}^2 = 9, c_2 = 3.$$

Thus,

$$N_\delta(d) = T_\delta(d^2, -3d, 9, 3).$$

A consequence of the GYZ formula

Recall the Göttsche-Yau-Zaslow's formula

$$\sum_{\delta \geq 0} T_\delta(x,y,z,w)(DG_2(q))^\delta = \frac{(DG_2(q)/q)^{\frac{z+w}{12}+\frac{x-y}{2}} B_1(q)^z B_2(q)^y}{(\Delta(q) D^2 G_2(q)/q^2)^{\frac{z+w}{24}}},$$

Proposition (Göttsche)**.** *If we form the generating function*

$$\mathcal{N}(t) := \sum_{\delta \geq 0} T_\delta(w,x,y,z) t^\delta,$$

and set $\mathcal{Q}(t) := \log \mathcal{N}(t)$, *then*

$$\mathcal{Q}(t) = wA_1(t) + xA_2(t) + yA_3(t) + zA_4(t).$$

for some $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[t]]$.

In other words, $Q_\delta(w,x,y,z) := [t^\delta]\mathcal{Q}(t)$ *is a* **linear** *function in* w, x, y, z.

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We call $Q_\delta(w,x,y,z)$ the *logarithmic version* of $T_\delta(w,x,y,z)$.

Logarithmic versions of Severi degrees

We let $Q^\delta(Y, \mathscr{L})$ be the *logarithmic version* of the generalized Severi degree $N^\delta(Y, \mathscr{L})$, that is,

$$\sum_{\delta \geq 1} Q^\delta(Y, \mathscr{L}) t^\delta = \log \left(\sum_{\delta \geq 0} N^\delta(Y, \mathscr{L}) t^\delta \right).$$

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Corollary. *For any fixed* δ*, there is a* **linear** *function* $Q_\delta(w, x, y, z)$ *(as we defined earlier) such that*

$$Q^\delta(Y, \mathscr{L}) = Q_\delta(\mathscr{L}^2, \mathscr{L} \cdot \mathscr{K}, \mathscr{K}^2, c_2)$$

whenever Y *is smooth and* $\mathscr{L}$ *is sufficiently ample, where* $\mathscr{K}$ *and* c_2 *are the canonical class and second Chern class of* Y*, respectively.*

Logarithmic versions of Severi degrees (cont'd)

Similarly, we let $Q^{d,\delta}$ be the *logarithmic version* of the classical Severi degree $N^{d,\delta}$, and $Q_\delta(d)$ the *logarithmic version* of the node polynomial $N_\delta(d)$.

Corollary. *For fixed δ and sufficiently large d, $Q^{d,\delta}$ is given by $Q_\delta(d)$*

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Corollary. *For fixed δ and sufficiently large d, $Q^{d,\delta}$ is given by $Q_\delta(d)$ which is a* **quadratic** *polynomial in d.*

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Proof. Recall that

$$N_\delta(d) = T_\delta(d^2, -3d, 9, 3).$$

Hence,

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Original Motivation Fomin-Mikhalkin's proof for the polynomiality of $N^{d,\delta}$ is combinatorial. Can we give a direct combinatorial proof for the above corollary?

PART II:

Computing Severi degrees via long-edge graphs

***Summary*:** We introduce long-edge graphs and Fomin-Mikhalkin's formula for computing classical Severi degrees and discuss our two main results, using which we give a combinatorial proof for the quadradicity of $Q^{d,\delta}$.

Some History

- Based on Mikhalkin's work, Brugallé and Mikhalkin gave an enumerative formula for the classical Severi degree $N^{d,\delta}$ in terms of "(marked) labeled floor diagrams". (2007-2008)

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- Block, Colley and Kennedy considered the logarithmic version of a special **single variable** function associated to long-edge graphs which appeared in Fomin-Mikhalkin's formula, and conjectured it to be **linear**. They have since proved their conjecture. (2012-13)
- We consider a special **multivariate** function $P_\beta(G)$ associated to long-edge graphs G that generalizes BCK's function and its logarithmic version $\Phi_\beta(G)$, and prove that $\Phi_\beta(G)$ is always **linear**. (2013)

Long-edge graphs

Definition. A *long-edge graph* G is a graph (V, E) with a weight function ρ satisfying the following conditions:

a) The vertex set $V = \mathbb{N} = \{0, 1, 2, \dots\}$, and the edge set E is finite.

b) Multiple edges are allowed, but loops are not.

c) The weight function $\rho : E \to \mathbb{P}$ assigns a positive integer to each edge.

d) There are no *short edges*, i.e., there's no edges connecting i and $i + 1$ with weight 1.

We define the *multiplicity* of G to be

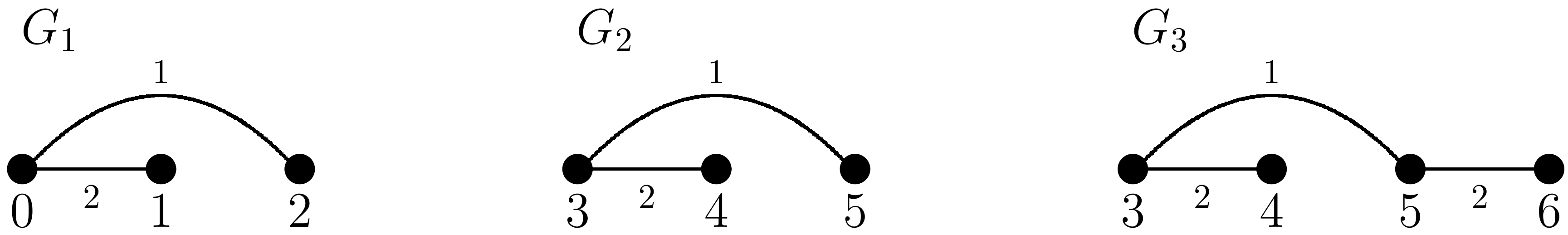
$$\mu(G) = \prod_{e \in E} (\rho(e))^2,$$

and the *cogenus* of G to be

$$\delta(G) = \sum_{e \in E} (l(e)\rho(e) - 1),$$

where for any $e = \{i, j\} \in E$ with $i < j$, we define $l(e) = j - i$.

Examples of long-edge graphs



$\mu(G_1) = \mu(G_2) = 2^2 \cdot 1^2 = 4, \quad \delta(G_1) = \delta(G_2) = (2 \cdot 1 - 1) + (1 \cdot 2 - 1) = 2,$

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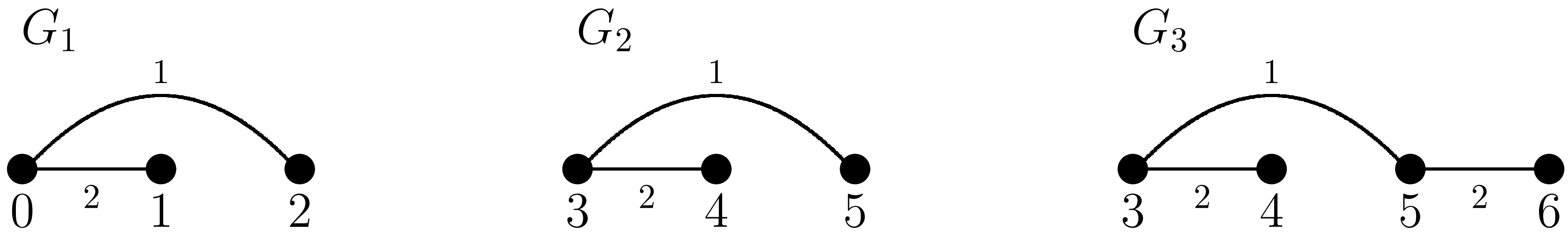

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Definitions by example

$G_2 = (G_1)_{(3)}$, since G_2 is obtained by shifting G_1 three units to the right.

$$\mathrm{maxv}(G_3) = 6, \quad \mathrm{minv}(G_3) = 3,$$

G_1 is a *template* because $\mathrm{minv}(G_1) = 0$ and we cannot "cut" G_1 into two nonempty subgraphs.

G_2 is a *shifted template*, and G_3 is **not** a shifted template.

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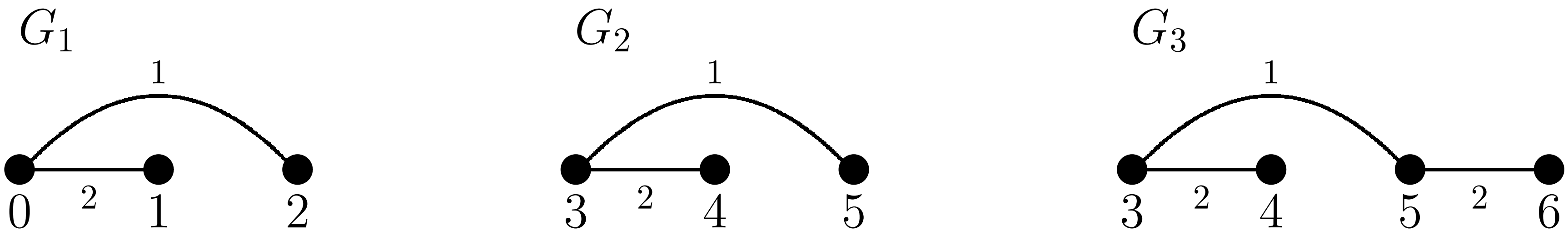

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Observation Any long-edge graph can be **decomposed** into shifted templates.

Concepts without detailed definitions

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_{M+1}) \in \mathbb{Z}_{\geq 0}^{M+1}$ (where $M \geq 0$).

- *β-allowable* and *strictly β-allowable*.

  Fact. A long-edge graph is **simultaneously** β-allowable and strictly β-allowable most of the time except for some "boundary" conditions.

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- *β-extended ordering*.

  Define

$$P_\beta(G) = \begin{cases} \#\ (\beta\text{-extended orderings of } G) & \text{if } G \text{ is } \beta\text{-allowable;} \\ 0 & \text{otherwise.} \end{cases}$$

  and

$$P_\beta^s(G) = \begin{cases} \#\ (\beta\text{-extended orderings of } G) & \text{if } G \text{ is strictly } \beta\text{-allowable;} \\ 0 & \text{otherwise.} \end{cases}$$

Fomin-Mikhalkin's formula

Theorem (Brugallé-Mikhalkin, Fomin-Mikhalkin)**.** *The classical Severi degree $N^{d,\delta}$ is given by*

$$N^{d,\delta} = \sum_{G:\ \delta(G)=\delta} \mu(G) P^{s}_{\mathbf{v}(d)}(G),$$

where

$$\mathbf{v}(d) := (0, 1, 2, \dots, d), \qquad \forall d \in \mathbb{Z}_{>0}.$$

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Logarithmic version

Recall that $Q^{d,\delta}$ is the logarithmic version $N^{d,\delta}$. We define $\Phi_\beta(G)$ and $\Phi^s_\beta(G)$ be the *logarithmic version* of $P_\beta(G)$ and $P^s_\beta(G)$, respectively. Then we obtain the **logarithmic version of Fomin-Mikhalkin's formula**:

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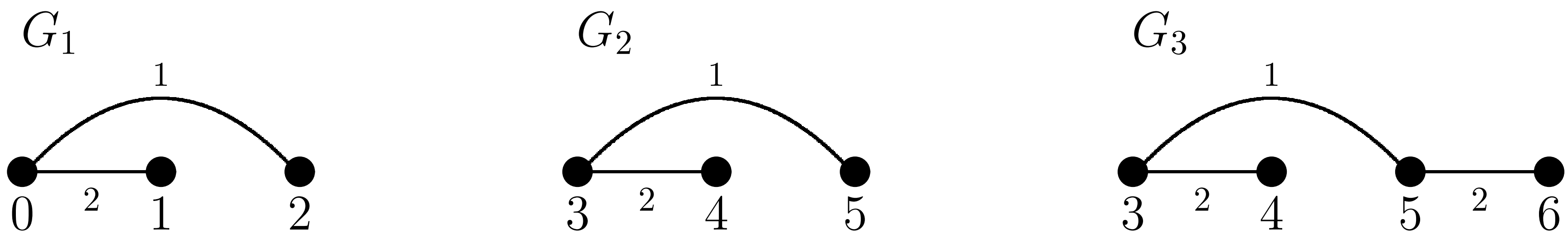
Our original motivation was to give a combinatorial proof for the result that $Q^{d,\delta}$ is given by **quadratic** polynomial, for sufficiently large d.

The Vanishing Lemma

Recall that among the three graphs in the figure,



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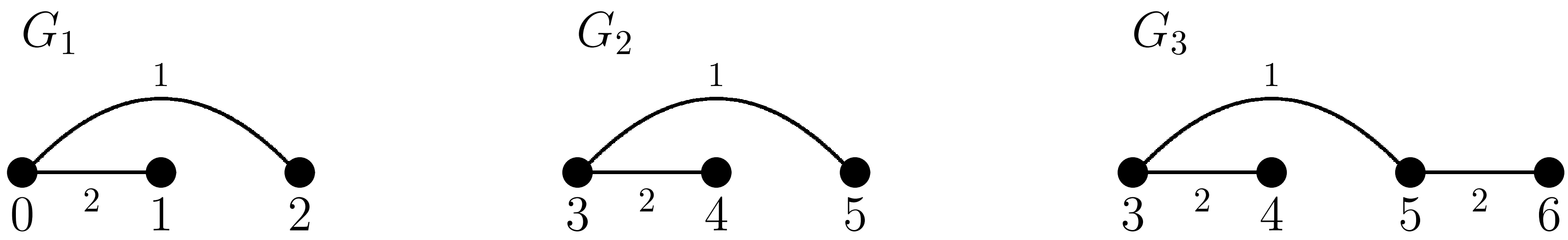

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Corollary (Block-Colley-Kennedy, L.)**.** *Suppose G is not a shifted template. Then* $\Phi^s_{\mathbf{v}(d)}(G) = 0$.

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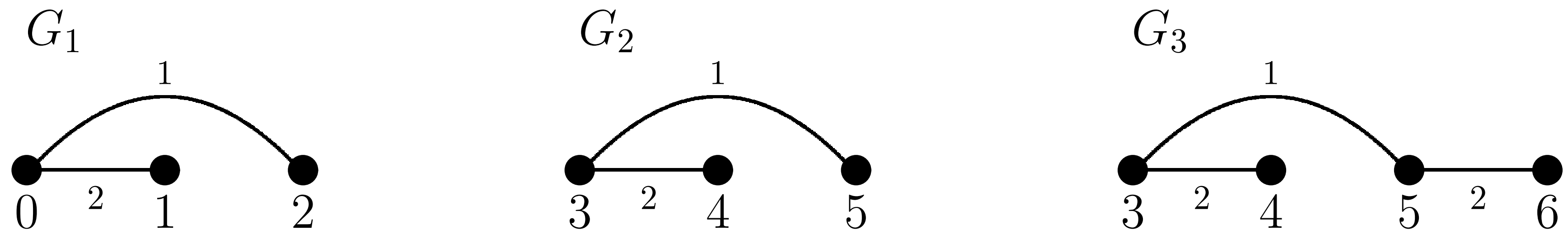

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Corollary (Block-Colley-Kennedy, L.)**.** *Suppose G is not a shifted template. Then* $\Phi^s_{\mathbf{v}(d)}(G) = 0$.

Applying the corollary, we get

$$Q^{d,\delta} = \sum_{G:\ \delta(G)=\delta} \mu(G)\Phi^s_{\mathbf{v}(d)}(G) = \sum_{\text{template } \Gamma:\ \delta(\Gamma)=\delta} \mu(\Gamma) \sum_{k\geq 0} \Phi^s_{\mathbf{v}(d)}\left(\Gamma_{(k)}\right),$$

The Linearity Theorem

Theorem (L.)**.** *Suppose* G *is a long-edge graph satisfying* $\text{maxv}(G) \leq M+1$. *Then for any sufficiently large* $\beta = (\beta_1, \dots, \beta_{M+1})$ *(depending on* G*), the values of* $\Phi_\beta(G)$ *are given by a* **linear** *multivariate function in* β.

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Corollary (Block-Colley-Kennedy, L.). *Suppose G is a long-edge graph. Then for sufficiently large k (depending on G), and suffciently large d (depending on G and k), $\Phi_{\mathbf{v}(d)}(G_{(k)})$ is a* **linear** *function in k.*

Quadraticity of $Q^{d,\delta}$

Sketch of Proof. We already show

$$Q^{d,\delta} = \sum_{\text{template } \Gamma:\ \delta(\Gamma)=\delta} \mu(\Gamma) \sum_{k \geq 0} \Phi^{s}_{\mathbf{v}(d)}\left(\Gamma_{(k)}\right).$$

Then the conclusion follows from the following points:

- There are finitely many templates of a given cogenus δ.

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- For fixed d, the second summation has finitely many terms. In fact, we were able to show that the second summation becomes

$$\sum_{k=0}^{d+\epsilon_1(\Gamma)-l(\Gamma)} \Phi^s_{\mathbf{v}(d)}\left(\Gamma_{(k)}\right) = \sum_{k=1}^{d+\epsilon_1(\Gamma)-l(\Gamma)} \Phi_{\mathbf{v}(d)}\left(\Gamma_{(k)}\right).$$

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- There are finitely many templates of a given cogenus δ.
- For fixed d, the second summation has finitely many terms. In fact, we were able to show that the second summation becomes

$$\sum_{k=0}^{d+\epsilon_1(\Gamma)-l(\Gamma)} \Phi^s_{\mathbf{v}(d)} \left(\Gamma_{(k)}\right) = \sum_{k=1}^{d+\epsilon_1(\Gamma)-l(\Gamma)} \Phi_{\mathbf{v}(d)} \left(\Gamma_{(k)}\right).$$

- It follows from the linearity corollary that except for first several terms, all other terms are a linear function in k. □

We can do more

- Recover the threshold bound $d \geq \delta$ for the polynomiality of $N^{d,\delta}$ obtained by Block.
- and ...

PART III:

Severi degrees on toric surfaces

Summary: We consider generalized Severi degrees on certain toric surfaces. By analyzing Ardila-Block's formula and applying the results from PART II, we obtain universality results that has close connection to Göttsche-Yau-Zaslow formula.

This is joint work with Brian Osserman.

Severi degrees $N^{\Delta,\delta}$

Recall that $N^\delta(Y, \mathscr{L})$ is the generalized Severi degree that counts the number of δ-nodal curves in $\mathscr{L}$ passing through $\dim|\mathscr{L}| - \delta$ points in general position, and $Q^\delta(Y, \mathscr{L})$ is its logarithmic version.

Given a lattice polygon Δ, let $Y(\Delta)$ be associated toric surface, and $\mathscr{L}(\Delta)$ be the line bundle, and set

$$N^{\Delta,\delta} := N^\delta(Y(\Delta), \mathscr{L}(\Delta)), \quad \text{and} \quad Q^{\Delta,\delta} := Q^\delta(Y(\Delta), \mathscr{L}(\Delta)).$$

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$$N^{\Delta,\delta} := N^\delta(Y(\Delta), \mathscr{L}(\Delta)), \quad \text{and} \quad Q^{\Delta,\delta} := Q^\delta(Y(\Delta), \mathscr{L}(\Delta)).$$

Recall that Fomin-Mikhalkin's formula for $N^{d,\delta}$ was derived from Brugallé-Mikhalkin's enumerative formula for Severi degrees using labeled floor diagrams.

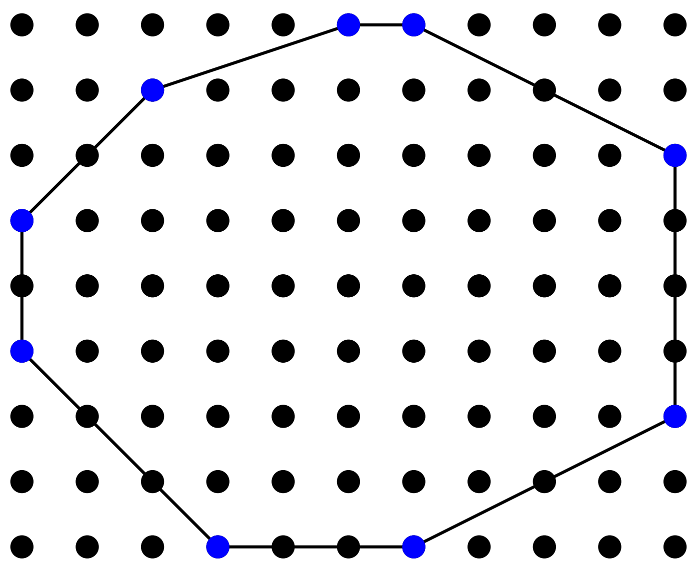
In fact, the formula introduced by Brugallé and Mikhalkin works **not only** for $N^{d,\delta}$, **but also** for Severi degrees $N^{\Delta,\delta}$ arising from *h-transverse* polygons Δ.

h-transverse polygon

Definition. A polygon Δ is *h-transverse* if all its normal vectors have infinite or integer slope.

If v is a vertex of Δ, we define $\det(v)$ to be $|\det(w_1, w_2)|$, where w_1 and w_2 are primitive integer normal vectors to the edges adjacent to v.

Example



$$\det(v) = \left|\det \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\right| = 2 > 1$$

$\Longrightarrow$ singularity

The normals of the top and bottom edges have slopes ∞ and $-\infty$.
The normals of the four edges on the left have slopes $-3, -1, 0$ and 1.
The normals of the three edges on the right have slopes $2, 0$ and -2.

Ardila-Block's work

In parallel to Fomin-Mikhalkin's work, Ardila and Block reformulate Brugallé-Mikhalkin's formula for $N^{\Delta,\delta}$ where Δ is an h-transverse polygon, and obtain polynomiality result.

Theorem (Brugallé-Mikhalkin, Ardila-Block)**.** *For any h-transverse polygon Δ and any $\delta \geq 0$, the Severi degree $N^{\Delta,\delta}$ is given by*

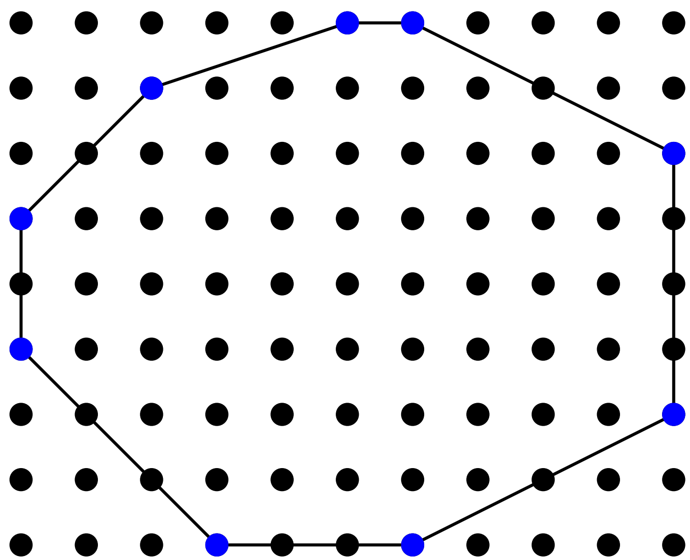
$$N^{\Delta,\delta} = \sum_{\Delta'} \sum_{G} \mu(G) P^{s}_{\beta(\Delta')}(G),$$

where the first summation is over all "reorderings" Δ' of Δ satisfying $\delta(\Delta') \leq \delta$, and the second summation is over all long-edge graphs G with $\delta(G) = \delta - \delta(\Delta')$.

Ardila-Block's work (cont'd)

Ardila and Block encode each h-transverse polygon Δ with two vectors $\mathbf{c}$ and $\mathbf{d}$.

Example



Slope vector:
$\mathbf{c} = ((2, 0, -2), (-3, -1, 0, 1))$

Edge length vector:
$\mathbf{d} = (1, (2, 4, 2), (1, 2, 2, 3))$

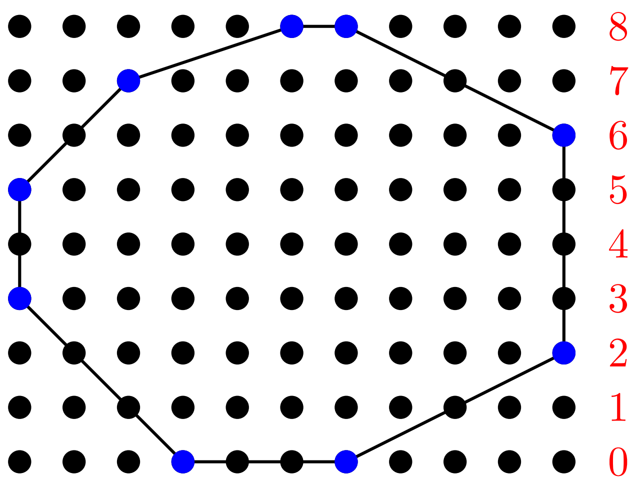
Write

$$\Delta = \Delta(\mathbf{c}, \mathbf{d}).$$

Ardila-Block's work (cont'd)

Theorem (Ardila-Block)**.** *Fixing δ and the number of edges on the left and right of Δ.*

- *For fixed* $\mathbf{c}$, *the number* $N^{\Delta,\delta}$ *is given by a* **polynomial** *in* $\mathbf{d}$ *for any choice of* $\mathbf{d}$ *such that the heights of vertices of* $\Delta(\mathbf{c},\mathbf{d})$ *are sufficiently spread out relative to* δ.
- *The number* $N^{\Delta,\delta}$ *is given by a* **polynomial** *in* $\mathbf{c}$ *and* $\mathbf{d}$ *for any* $\mathbf{c}$ *that is sufficiently spread out, any choice of* $\mathbf{d}$ *such that the heights of vertices of* $\Delta(\mathbf{c},\mathbf{d})$ *are sufficiently spread out relative to* δ.



Tzeng's theorem

Recall Göttsche's conjecture/Tzeng's theorem:

(i) For every fixed δ, there exists a **universal polynomial** $T_\delta(w, x, y, z)$ of degree δ such that

$$N^\delta(Y, \mathscr{L}) = T_\delta(\mathscr{L}^2, \mathscr{L} \cdot \mathscr{K}, \mathscr{K}^2, c_2)$$

whenever Y is smooth and $\mathscr{L}$ is $(5\delta - 1)$-ample, where $\mathscr{K}$ and c_2 are the canonical class and second Chern class of Y, respectively.

(ii) Moreover, there exist power series $B_1(q)$ and $B_2(q)$ such that

$$\sum_{\delta \geq 0} T_\delta(x, y, z, w)(DG_2(q))^\delta = \frac{(DG_2(q)/q)^{\frac{z+w}{12} + \frac{x-y}{2}} B_1(q)^z B_2(q)^y}{(\Delta(q) D^2 G_2(q)/q^2)^{\frac{z+w}{24}}},$$

where $G_2(q) = -\frac{1}{24} + \sum_{n>0} \left(\sum_{d|n} d \right) q^n$ is the second Eisenstein series, $D = q\frac{d}{dq}$ and $\Delta(q) = q \prod_{k>0}(1 - q^k)^{24}$ is the modular discriminant. The above formula is known as the *Göttsche-Yau-Zaslow formula*.

Ardila-Block's work vs Tzeng's theorem

(i) Advantage: Treats many **singular** surfaces when Tzeng's theorem **only** covers **smooth surfaces**.

(ii) Disadvantage: The **universality** is **not** nearly as strong:

– need to fix the number of edges on the left and right;

– infinite slopes are treated differently;

– the number of variables grows with the number of edges;

– no results like the Göttsche-Yau-Zaslow formula.

Strongly h-transverse

Recall that Ardila-Block's formula

$$N^{\Delta,\delta} = \sum_{\Delta'} \sum_{G} \mu(G) P^s_{\beta(\Delta')}(G),$$

is very similar to Fomin-Mikhalkin's formula. Thus, naturally we consider the logarithmic version of it:

$$Q^{\Delta,\delta} = \sum_{\Delta'} \sum_{G} \mu(G) \Phi^s_{\beta(\Delta')}(G),$$

By applying the Vanishing Lemma and the Linearity Theorem, we are able to give a formula for $Q^{\Delta,\delta}$.

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Definition. We say an h-transverse polygon Δ is *strongly h-transverse* if either there is a non-zero horizontal edge at the top of Δ, or the vertex v at the top has $\det(v) \in \{1, 2\}$, and the same holds for the bottom of Δ.

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It turns out that an h-transverse polygon Δ is strongly h-transverse if and only if $Y(\Delta)$ is Gorenstein.

Main result

Recall the following corollary to Tzeng's theorem:

Corollary. *For any fixed* δ*, there is a* **linear** *function* $Q_\delta(w, x, y, z)$ *such that*

$$Q^\delta(Y, \mathscr{L}) = Q_\delta(\mathscr{L}^2, \mathscr{L} \cdot \mathscr{K}, \mathscr{K}^2, c_2)$$

whenever Y *is smooth and* $\mathscr{L}$ *is sufficiently ample, where* $\mathscr{K}$ *and* c_2 *are the canonical class and second Chern class of* Y*, respectively.*

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Theorem (L.-Osserman)**.** *Fix* $\delta \geq 1$*. Then there exist* **constants** $E(\delta)$ *and* $E_i(\delta)$ *for* $i = 1, \ldots, \delta - 1$ *such that if* Δ *is a strongly h-transverse polygon with all edges having length at least* δ*, then*

$$Q^{\Delta,\delta} = Q_\delta(\mathscr{L}(\Delta)^2, \mathscr{L}(\Delta) \cdot \mathscr{K}, \mathscr{K}^2, \tilde{c}_2) + E(\delta)S + \sum_{i=1}^{\delta-1} E_i(\delta) S_i,$$

where $\mathscr{K}$ *is the canonical line bundle on* $Y(\Delta)$*,* S_i *is the number of singularities of* $Y(\Delta)$ *of Milnor number* i*,* $\tilde{c}_2 = c_2(Y(\Delta)) + \sum_{i \geq 1} i S_i$*, and* $S = \sum_{i \geq 1}(i+1)S_i$*.*

Connection to Tzeng's Theorem

Theorem (L.-Osserman)**.** *We have the following:*

(i) For every fixed δ*, there exists a* **universal polynomial** $T_\delta(w, x, y, z; s, s_1, \ldots, s_{\delta-1})$ *such that*

$$N^{\Delta,\delta} = T_\delta(\mathscr{L}^2, \mathscr{L}\cdot\mathscr{K}, \mathscr{K}^2, \tilde{c}_2; S, S_1, \ldots, S_{\delta-1})$$

whenever Δ *is a strongly h-transverse* polygon *with all edges having length at least* δ*.*

(ii) Moreover,

$$\sum_{\delta\geq 0} T_\delta(\mathscr{L}^2, \mathscr{L}\cdot\mathscr{K}, \mathscr{K}^2, \tilde{c}_2; S, S_1, S_2, \ldots)(DG_2(\tau))^\delta$$

$$= \frac{(DG_2(\tau)/q)^{\chi(\mathscr{L})} B_1(q)^{\mathscr{K}^2} B_2(q)^{\mathscr{L}\cdot\mathscr{K}}}{(\Delta(\tau)D^2G_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}} \mathcal{P}(q)^{-S} \prod_{i\geq 2} \mathcal{P}\left(q^i\right)^{S_{i-1}},$$

where $\mathcal{P}(x) = \sum_{n\geq 0} p(n)x^n$ *is the generating function for partitions.*

Formulas for $B_1(q)$ and $B_2(q)$

Corollary. *we have*

$$B_1(q) = (\mathcal{P}(q))^{-1} \cdot \exp\left(- \sum_{\delta \geq 1} D(\delta) \left(DG_2(q)\right)^{\delta} \right),$$

$$B_2(q) = \exp\left(\sum_{\delta \geq 1} (A(\delta) - L(\delta)) \left(DG_2(q)\right)^{\delta} \right).$$

Here

$$A(\delta) = \frac{1}{2} \sum \mu(\Gamma) \zeta^0(\Gamma),$$

$$L(\delta) := -\frac{1}{2} \sum \mu(\Gamma) \zeta^0(\Gamma)(\ell(\Gamma) - \epsilon_0(\Gamma) - \epsilon_1(\Gamma)),$$

$$D(\delta) := -\sum \mu(\Gamma) \left(\zeta^2(\Gamma) + \zeta^1(\Gamma)(1 - \epsilon_0(\Gamma)) \right),$$

where all summations are over templates of cogenus δ.