Combinatorics of nested Braid fan

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Combinatorics Seminar

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This is joint work with Federico Castillo.

Outline

- Preliminary: Basic definitions of polytopes
- Motivation: Permutohedra and the Braid fan
- Nested permutohedra and the nested Braid fan (joint work with Castillo).

PART I:

Preliminary

Polytopes

Let V be a d-dimensional real vector space (or affine space), and V^* the dual space of V consisting of all linear functionals on V. This defines a perfect pairing $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ by $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = \boldsymbol{a}(\boldsymbol{x})$, for $\boldsymbol{a} \in V^*$ and $\boldsymbol{x} \in V$.

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Definition (Convex-hull defintiion)**.** A *polytope* $P \subset V$ is the convex hull of finite many points $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ in V:

$$P := \operatorname{conv}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) = \left\{ \sum_{i=1}^n \lambda_i \boldsymbol{v}_i \ : \ \text{all } \lambda_i \geq 0, \ \text{and } \sum_{i=1}^n \lambda_i = 1 \right\}.$$

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By the Minkowski-Weyl Theorem, we also have the following equivalent definition.

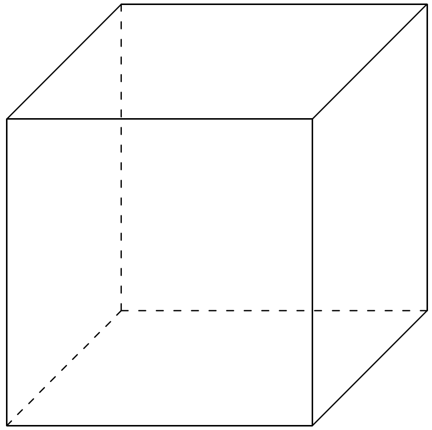
Definition (Inequality description)**.** A *polyhedron* $P \subset V$ is the solution set of a system of linear inequalities:

$$P = \{\boldsymbol{x} : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i, i \in I\},$$

where I is some indexing set, $\boldsymbol{a}_i \in V^*$ and $b_i \in \mathbb{R}$.

A *polytope* is a bounded polyhedron.

An example: Cube



A 3-dimensional cube defined as:

$$\mathrm{conv}\begin{pmatrix} (0,0,0),(0,0,1),(0,1,0),(0,1,1), \\ (1,0,0),(1,0,1),(1,1,0),(1,1,1) \end{pmatrix}$$

Alternatively, it can be defined by 6 inequalities:

$$\langle -\boldsymbol{e}_i, \boldsymbol{x}\rangle = -x_i \le 0, \quad \langle \boldsymbol{e}_i, \boldsymbol{x}\rangle = x_i \le 1, \quad i = 1,2,3$$

Faces

Definition. Let $\boldsymbol{u} \in V^*$. Define $c_{\boldsymbol{u}} := \max_{\boldsymbol{y} \in P} \langle \boldsymbol{u}, \boldsymbol{y} \rangle$. The set

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The *dimension* of a face is the dimension of its affine hull: $\dim(F) := \dim(\mathrm{aff}(F))$.

The faces of dimension $0, 1$, and $\dim(P) - 1$ are called *vertices, edges,* and *facets*.

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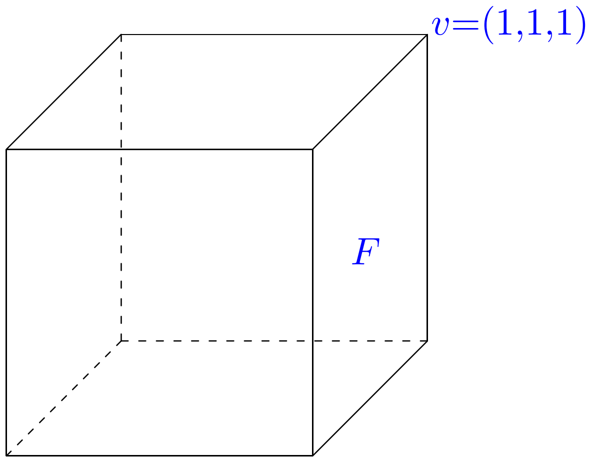

A 3-dimensional cube has:

8 vertices,

12 edges,

6 facets.

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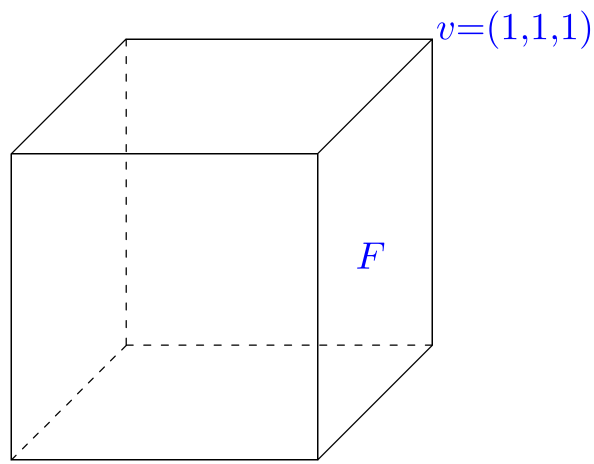

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i. The vertex $v = (1, 1, 1)$ can be obtained by choosing

(a) $\boldsymbol{u} = (1, 1, 1)$ and $c_{\boldsymbol{u}} = 3$, or

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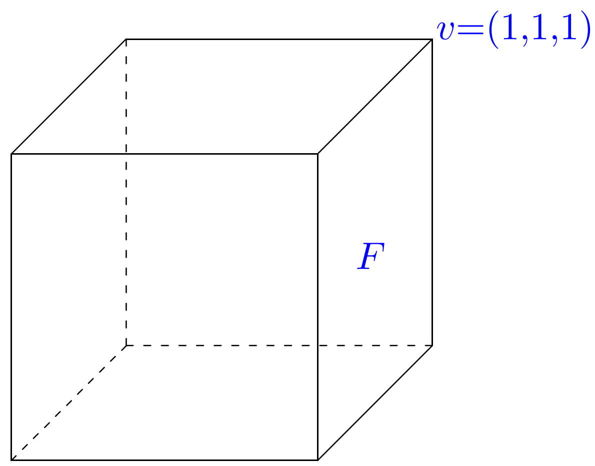

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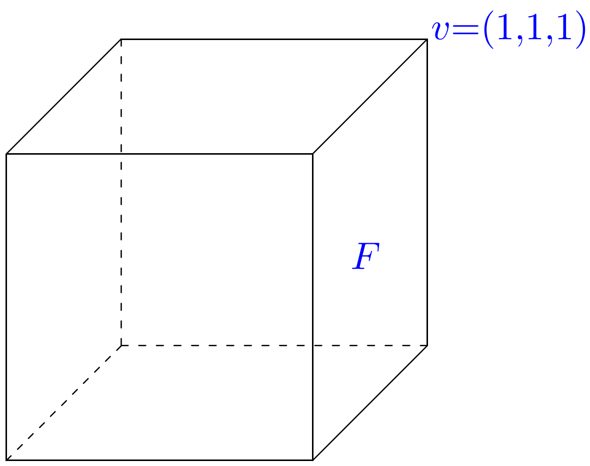

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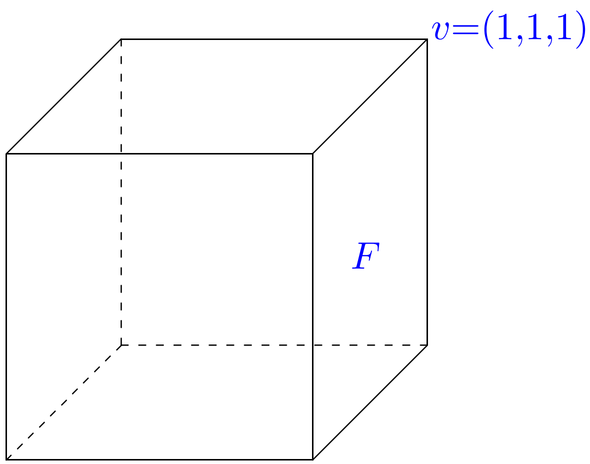

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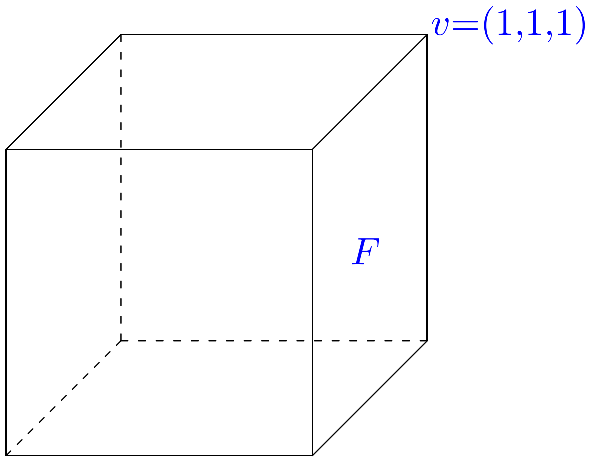

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Note that $\boldsymbol{e}_1$ is an outer normal vector of F.

Descriptions of polytopes

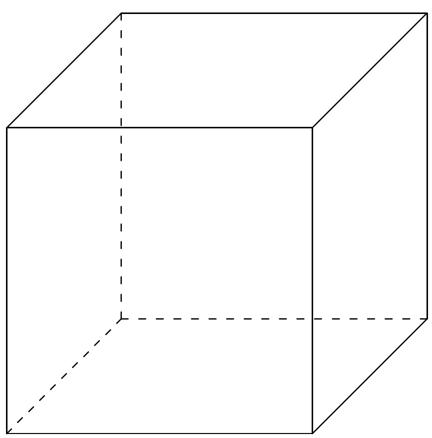
Suppose P is a full-dimensional polytope in V.

i. P can be described as the convex hull of its vertices: $P = \mathrm{conv}(\text{vert}(P))$.

ii. P can be described by an inequality description in the form of

$$\langle \boldsymbol{a}_F, \boldsymbol{x} \rangle \leq b_F, \quad F \in \text{facet}(P).$$

Here $\boldsymbol{a}_F$ is an outer normal vector of F.



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Definition. Given any face F of $P \subset V$, the *normal cone* of P at F is

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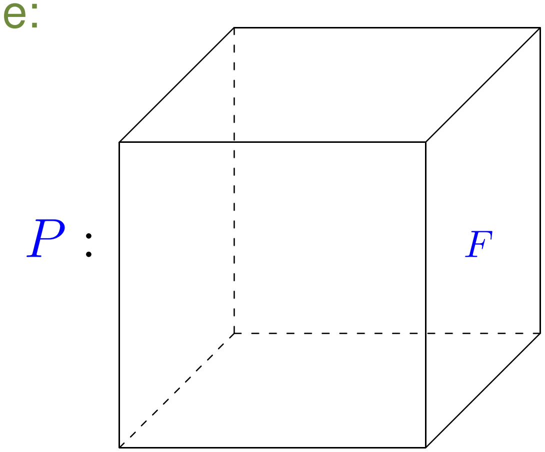

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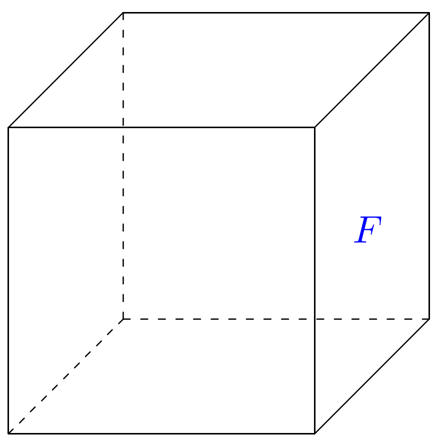

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FACT 1: If F is a *facet*, then $\mathrm{ncone}(F, P)$ is the cone spanned by its outer normal.

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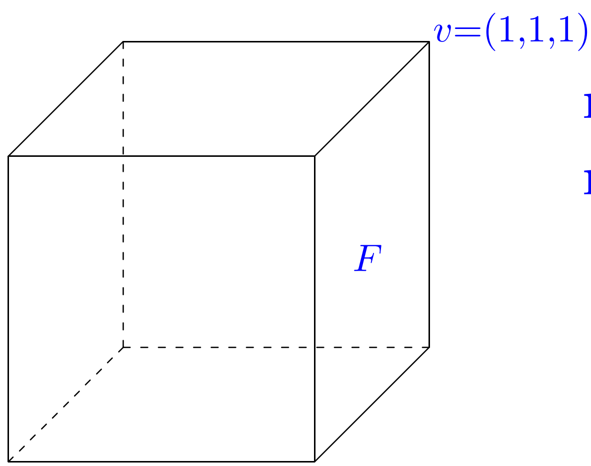

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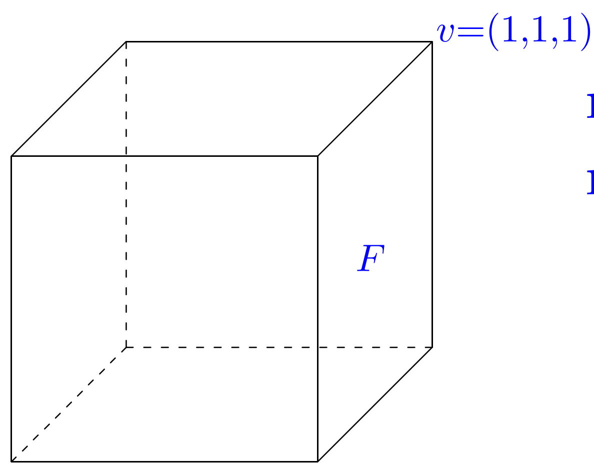

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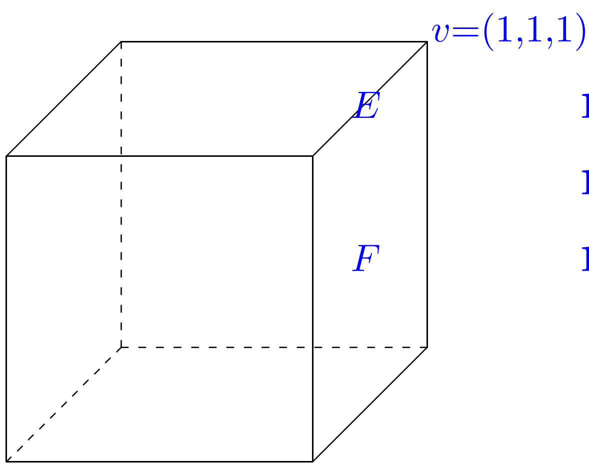

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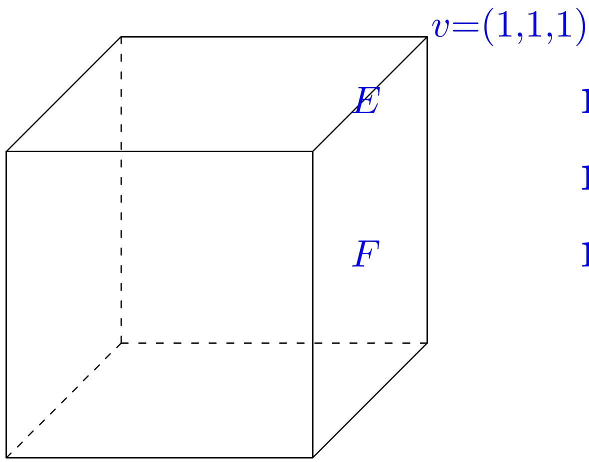

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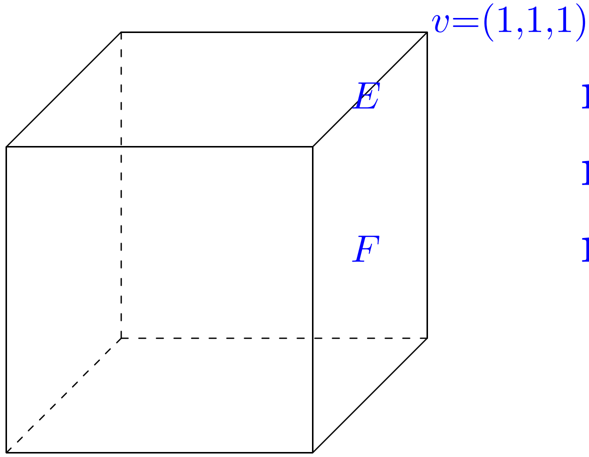
P : 

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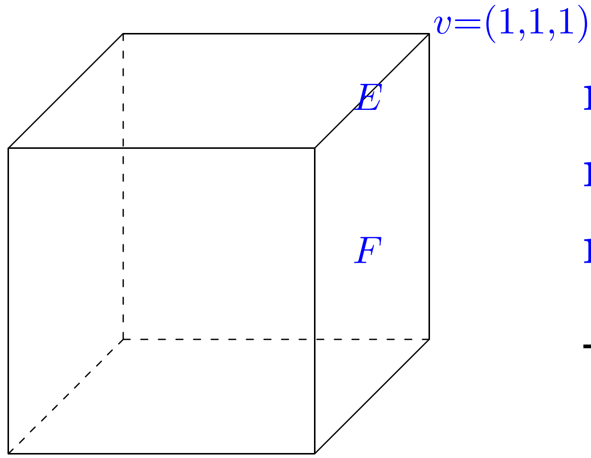

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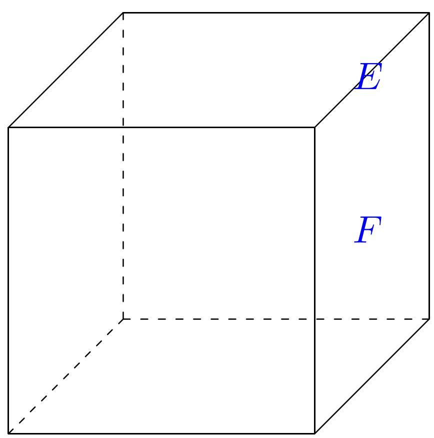

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In particular, vertices $\longleftrightarrow$ maximal cones

facets $\longleftrightarrow$ rays

PART II:

Motivation: Permutohedra and the Braid fan

For the rest of this talk, we have

$$V = \left\{ \boldsymbol{x} \in \mathbb{R}^{d+1} \ : \ \sum_{i=1}^{d+1} x_i = M \right\}$$

for some fixed M, so

$$V^* = \mathbb{R}^{d+1}/(1, 1, \ldots, 1)$$

.

Usual permutohedra

Definition. Given a strictly increasing sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_{d+1}) \in \mathbb{R}^{d+1}$, for any $\pi \in \mathfrak{S}_{d+1}$, we use the following notation:

$$v_\pi^{\boldsymbol{\alpha}} := \left(\alpha_{\pi(1)}, \alpha_{\pi(2)}, \cdots, \alpha_{\pi(d+1)}\right) = \sum_{i=1}^{d+1} \alpha_{\pi(i)} \boldsymbol{e}_i = \sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_{\pi^{-1}(i)}.$$

Then we define the *usual permutohedron*

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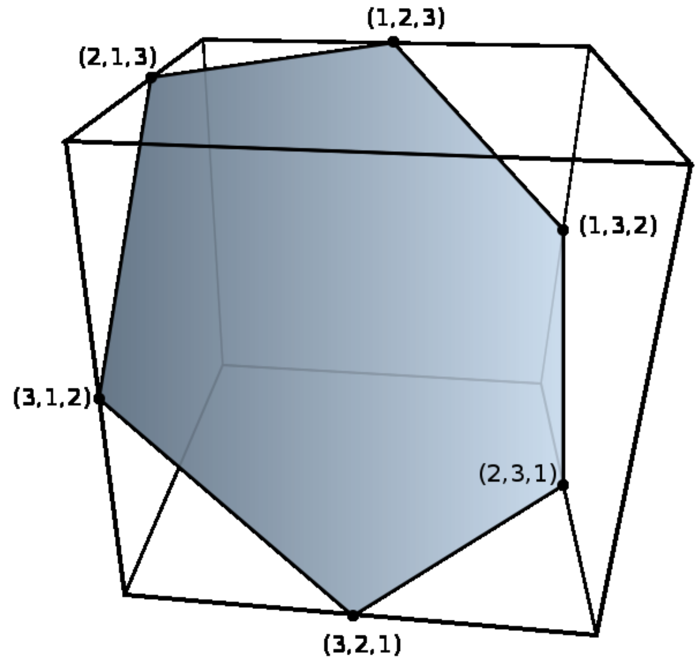
- If $\boldsymbol{\alpha} = (1, 2, \ldots, d, d+1)$, we obtain the *regular permutohedron* Π_d.

Example. Π_2:



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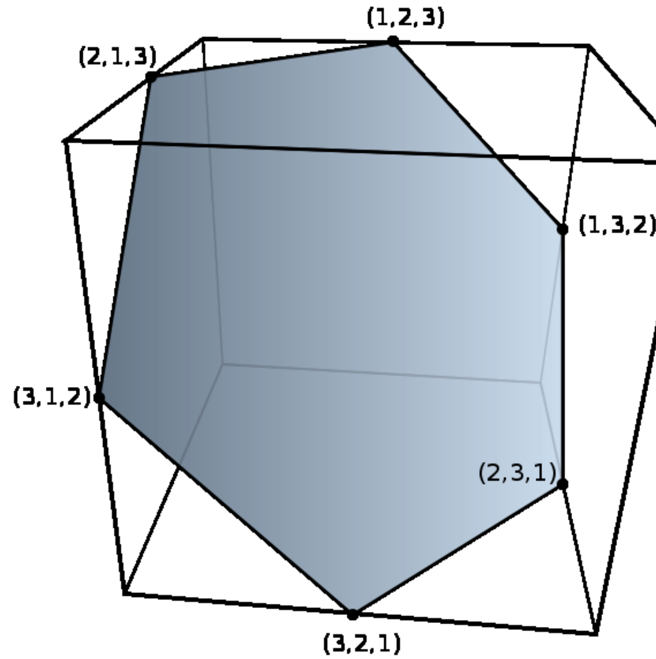

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- Any usual permutohedron $\mathrm{Perm}(\boldsymbol{\alpha})$ in $\mathbb{R}^{d+1}$ is d-dimensional, and so is full-dimensional in $V = \{\boldsymbol{x} \in \mathbb{R}^{d+1} \ : \ x_1 + \cdots + x_{d+1} = \alpha_1 + \cdots + \alpha_{d+1}\}$.

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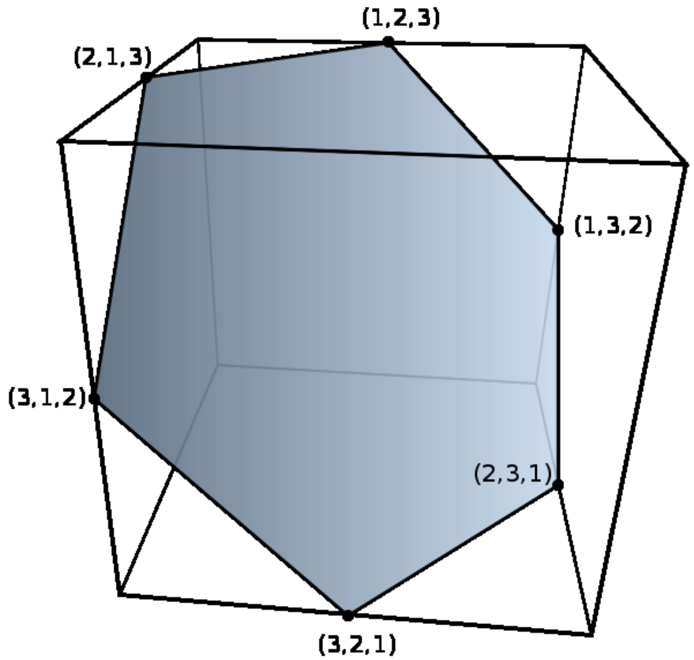

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- $v_\pi^{\boldsymbol{\alpha}} = \sum \alpha_i \boldsymbol{e}_{\pi^{-1}(i)}$ are vertices of $\mathrm{Perm}(\boldsymbol{\alpha})$.

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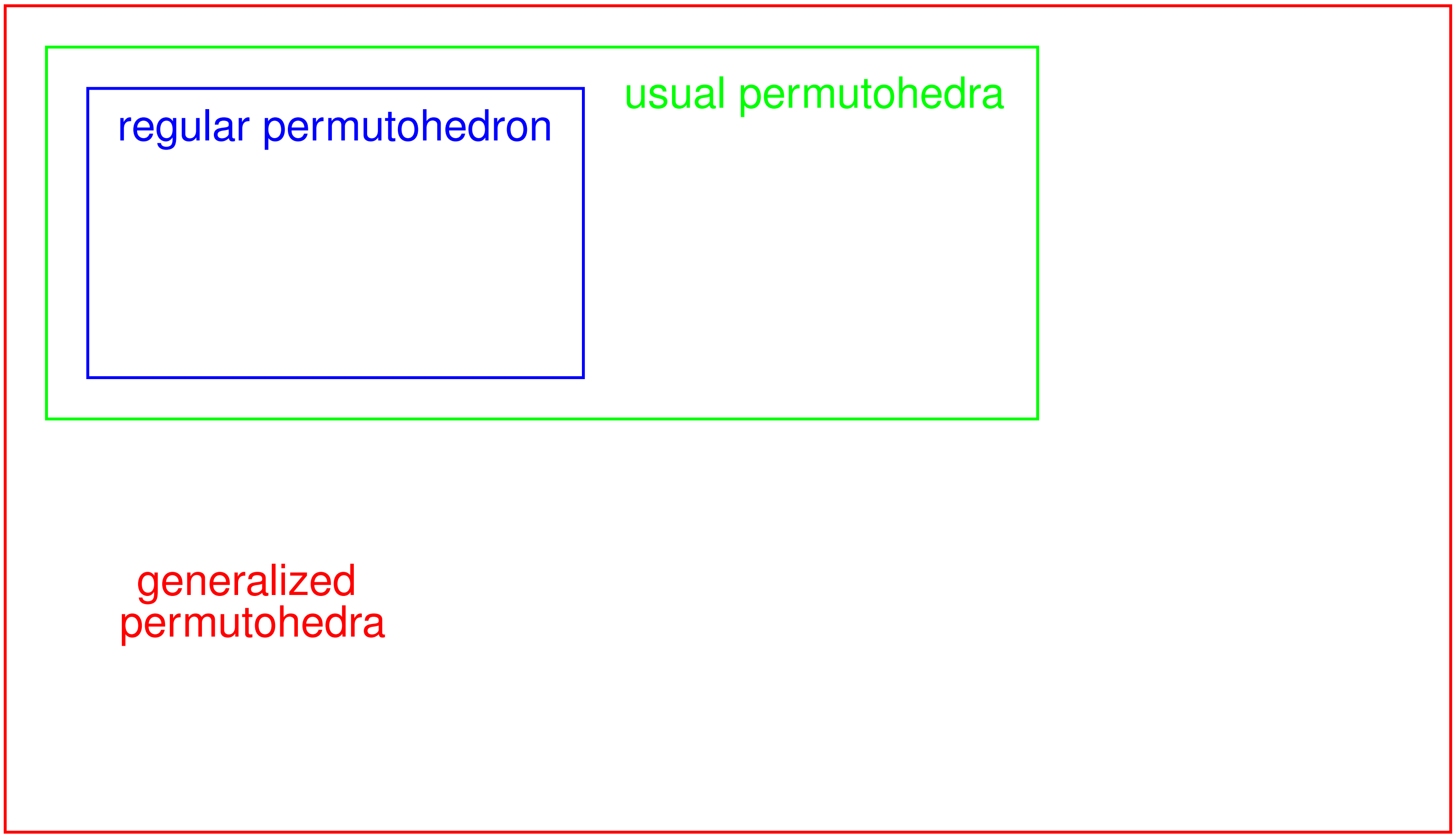

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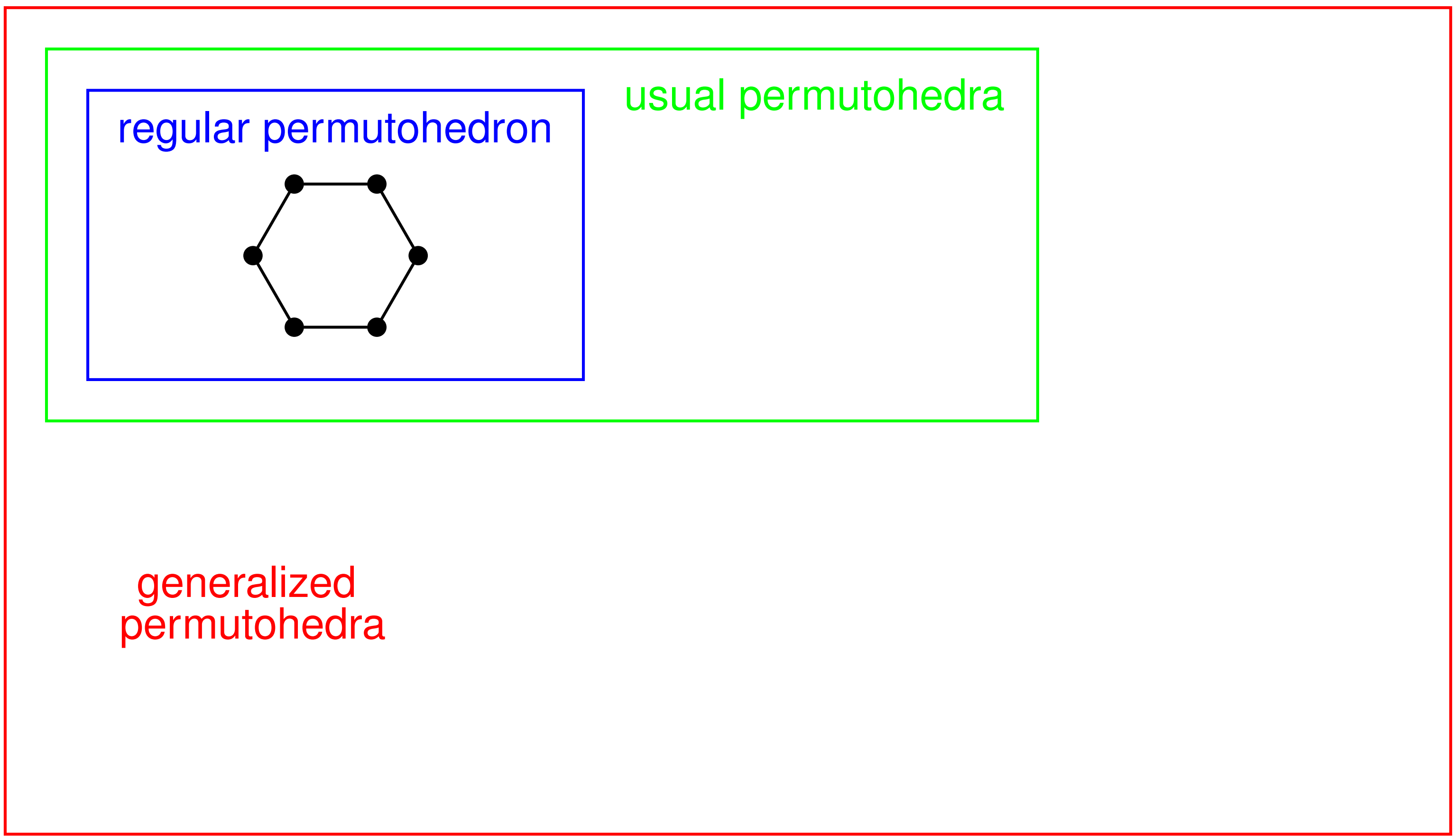

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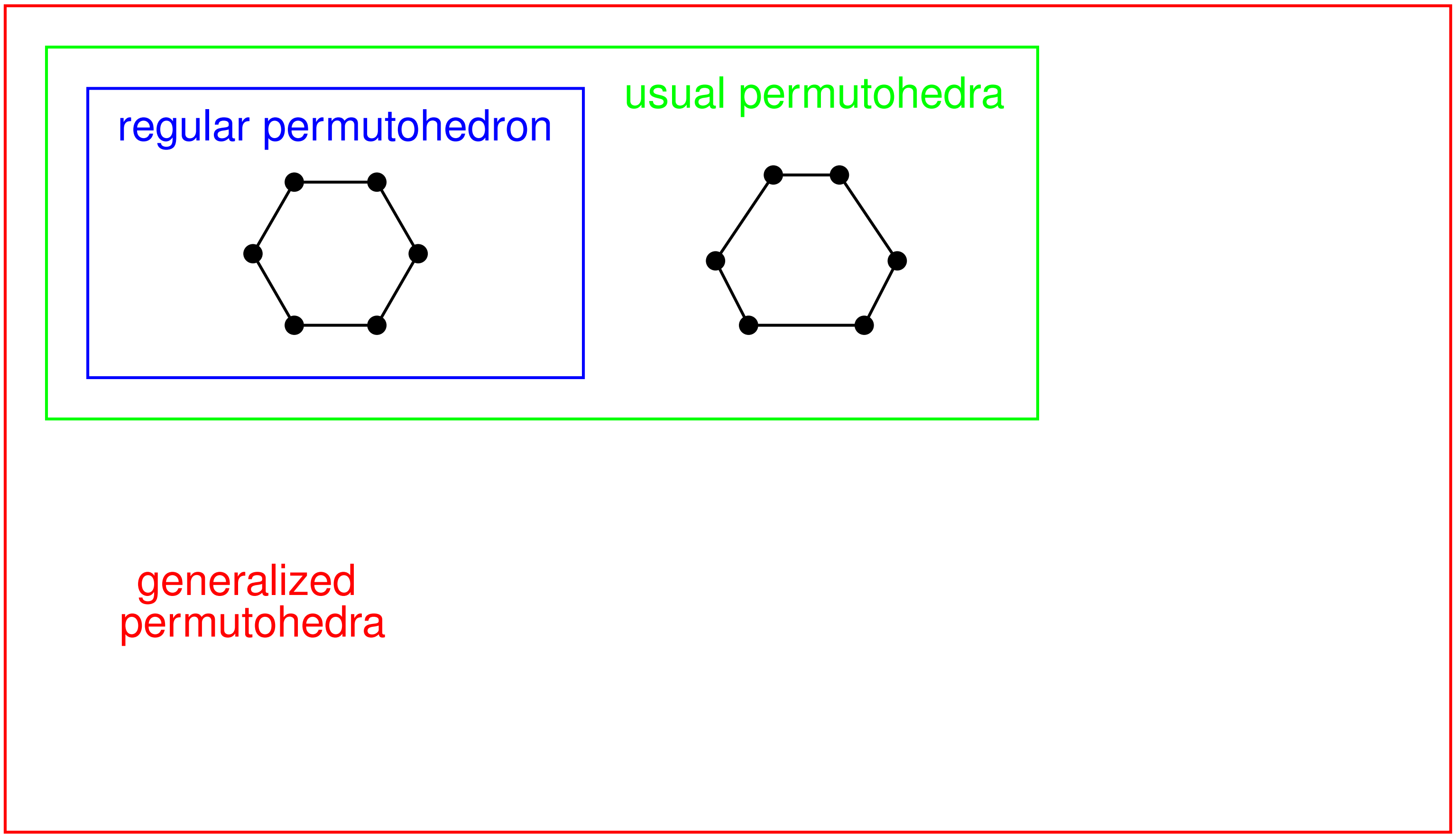

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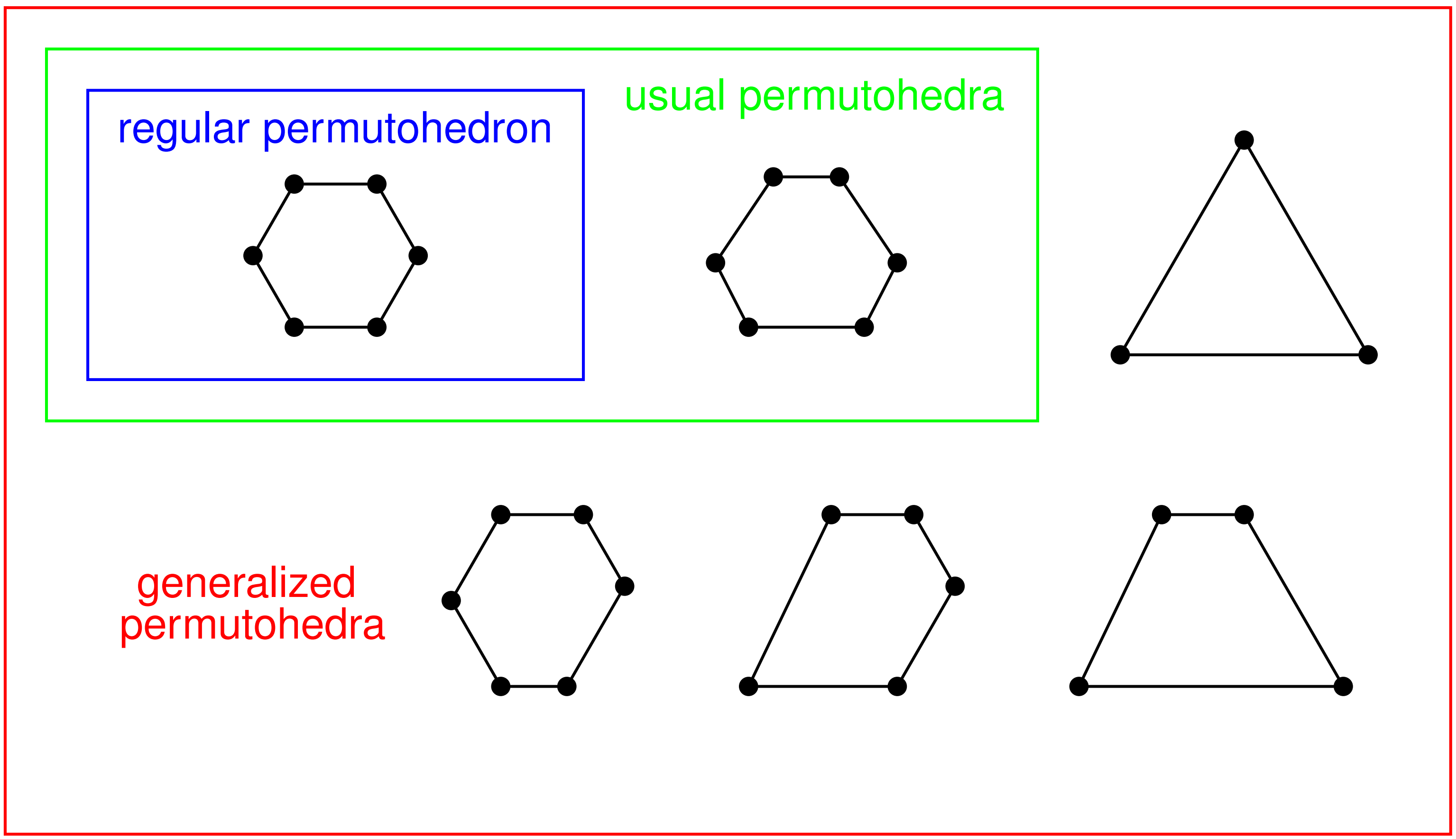
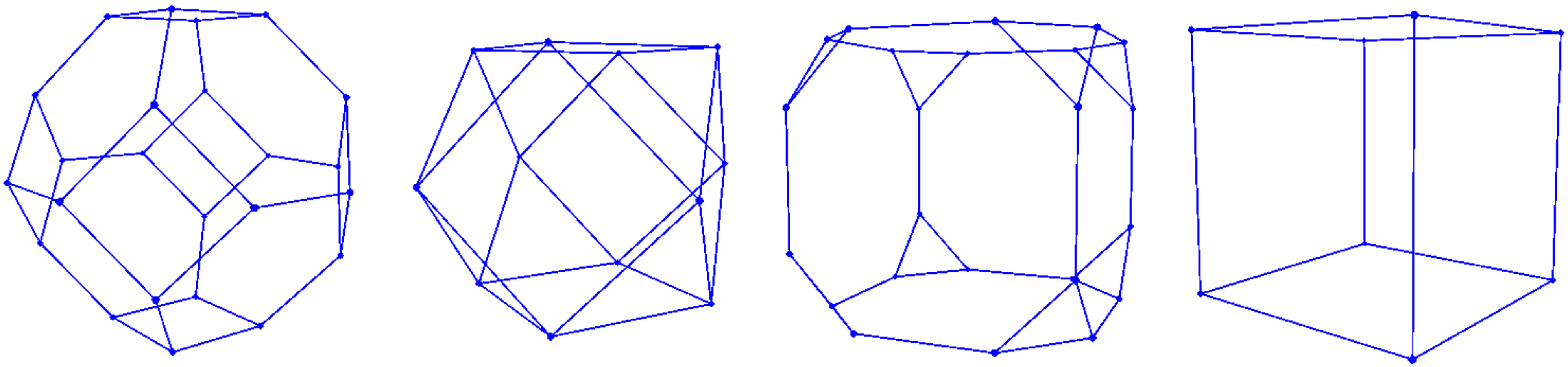


Nonexample

Example. Start with $P = \Pi_3 = \mathrm{Perm}((1, 2, 3, 4))$. By pushing all squares inward,



we obtain a 3-dimensional cube

$$Q = \mathrm{conv}(\mathrm{Perm}((1, 3, 3, 3)) \cup \mathrm{Perm}(2, 2, 2, 4))).$$

Q is *not* a generalized permutohedron.

Alternative definition

Let $V^* = \mathbb{R}^{d+1}/(1, 1, \ldots, 1)$. The *Braid fan* denoted by Br_d, is the complete fan in V^* given by the hyperplanes

$$x_i - x_j = 0 \quad \text{for all } i \neq j.$$

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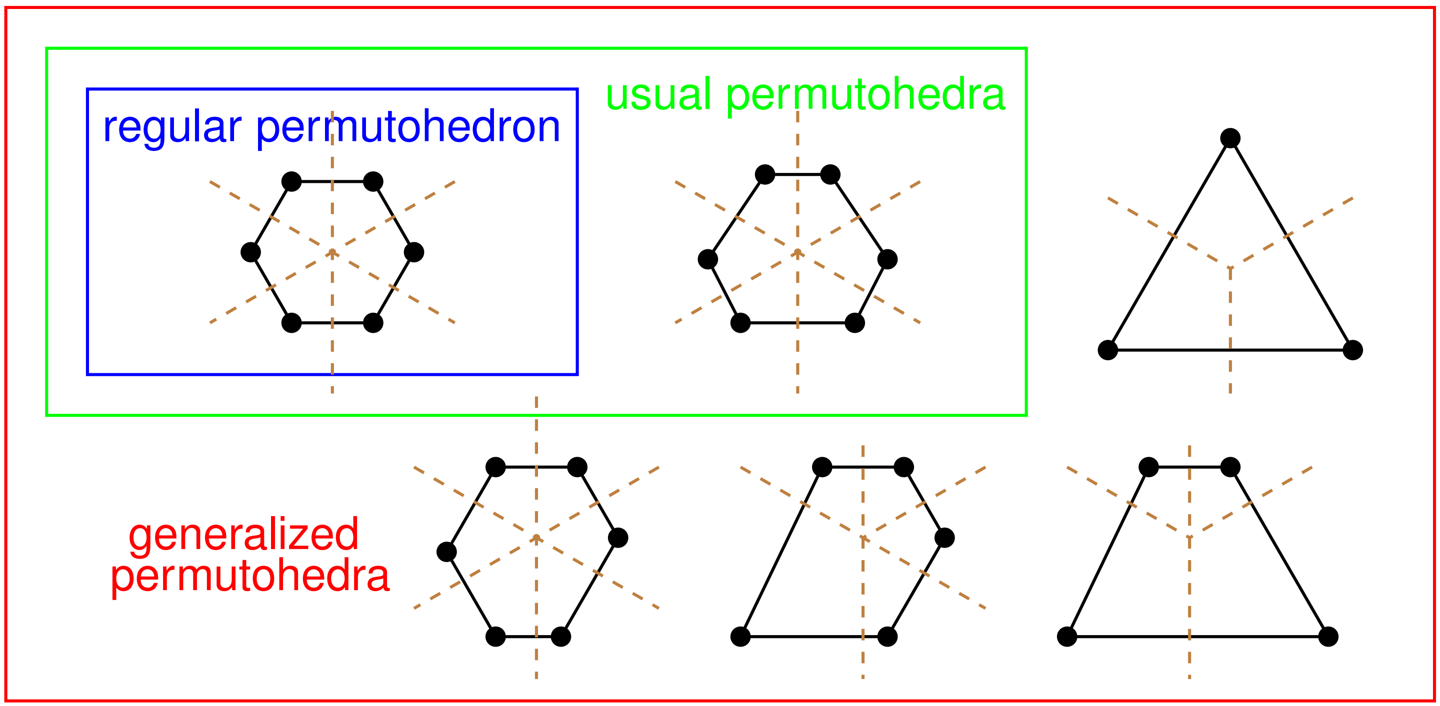

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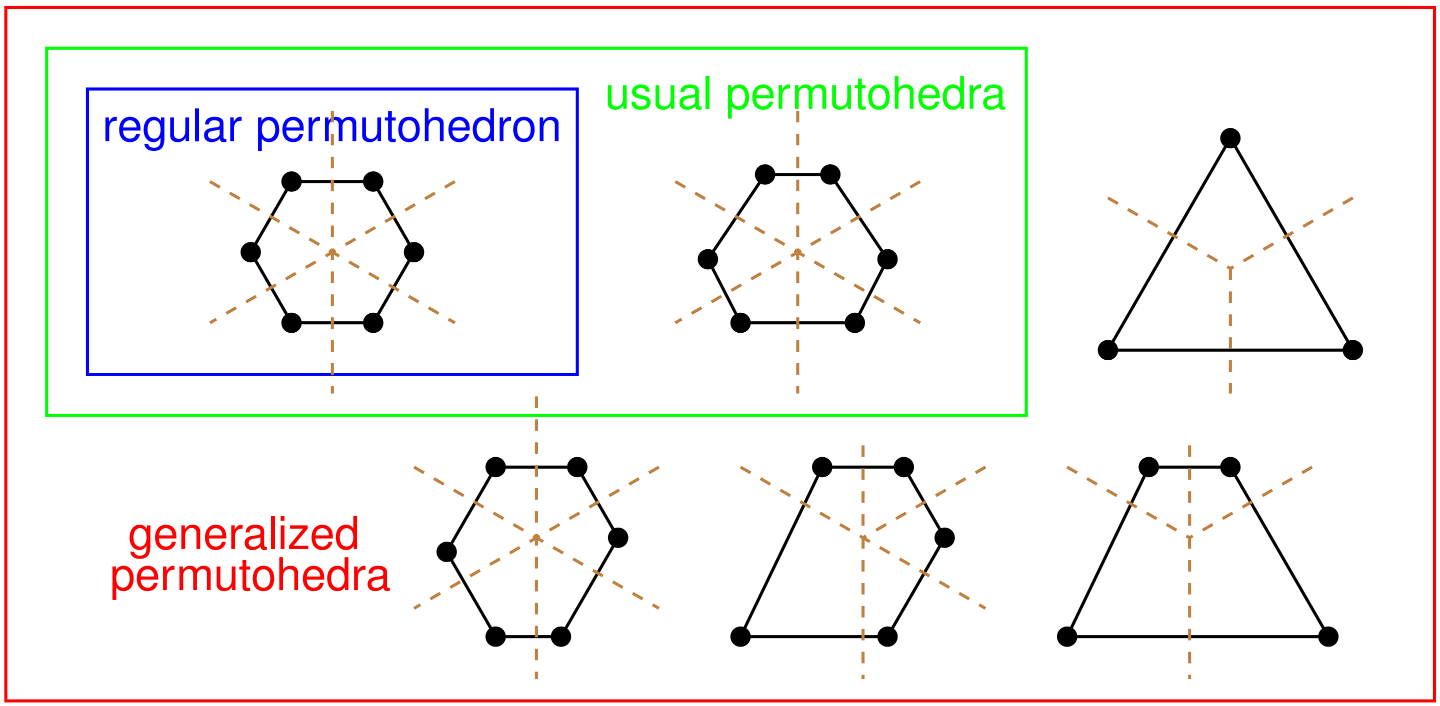


Fact. The normal fan of usual permutohedron in $\mathbb{R}^{d+1}$ is Br_d.

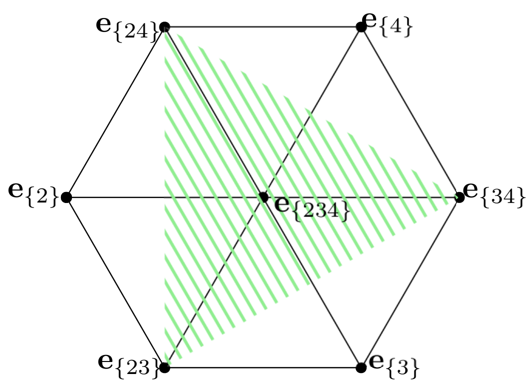
Nonexample

The alternative definition provides ways of verifying that a polytope is *not* a generalized permutohedron.

Recall our nonexample

$$Q = \mathrm{conv}(\mathrm{Perm}((1,3,3,3)) \cup \mathrm{Perm}(2,2,2,4))).$$

(i) Let σ be the normal cone of Q at the vertex $(1,3,3,3)$. It is *not* a coarsening of cones in Br_3.



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$$Q = \mathrm{conv}(\mathrm{Perm}((1,3,3,3)) \cup \mathrm{Perm}(2,2,2,4))).$$

(ii) The "walls" in Br_d are in the form of $x_i - x_j = 0$, which implies that edge directions of a generalized permutohedron are in the form of $\boldsymbol{e}_i - \boldsymbol{e}_j$.

But in Q, the vertices $(1,3,3,3)$ and $(2,4,4,4)$ form an edge whose direction is parallel to

$$(-1,1,1,-1) = \boldsymbol{e}_2 + \boldsymbol{e}_3 - \boldsymbol{e}_1 - \boldsymbol{e}_4.$$

Motivation

Question:

Can we extend the family of generalized permutohedra further to allow edge directions of the form $\boldsymbol{e}_i + \boldsymbol{e}_j - \boldsymbol{e}_k - \boldsymbol{e}_\ell$?

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Does not work! The combinatorics is not nice.

Final solution (joint work with F. Castillo):

Nested permotohedra and the nested Braid fan

PART III:

Nested permutohedra and the nested Braid fan

- Definition and answer to the motivating question
- Other properties of permutohedra and the Braid fan that can be generalized

Usual nested permutohedra

Definition (Informal). Replace each vertex of a usual permutohedron $\mathrm{Perm}(\boldsymbol{\alpha})$ by a smaller dimension permutohedron $\mathrm{Perm}(\boldsymbol{\beta})$ (in the correct orientation). We obtain the *usual nested permutohedron* $\mathrm{Perm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

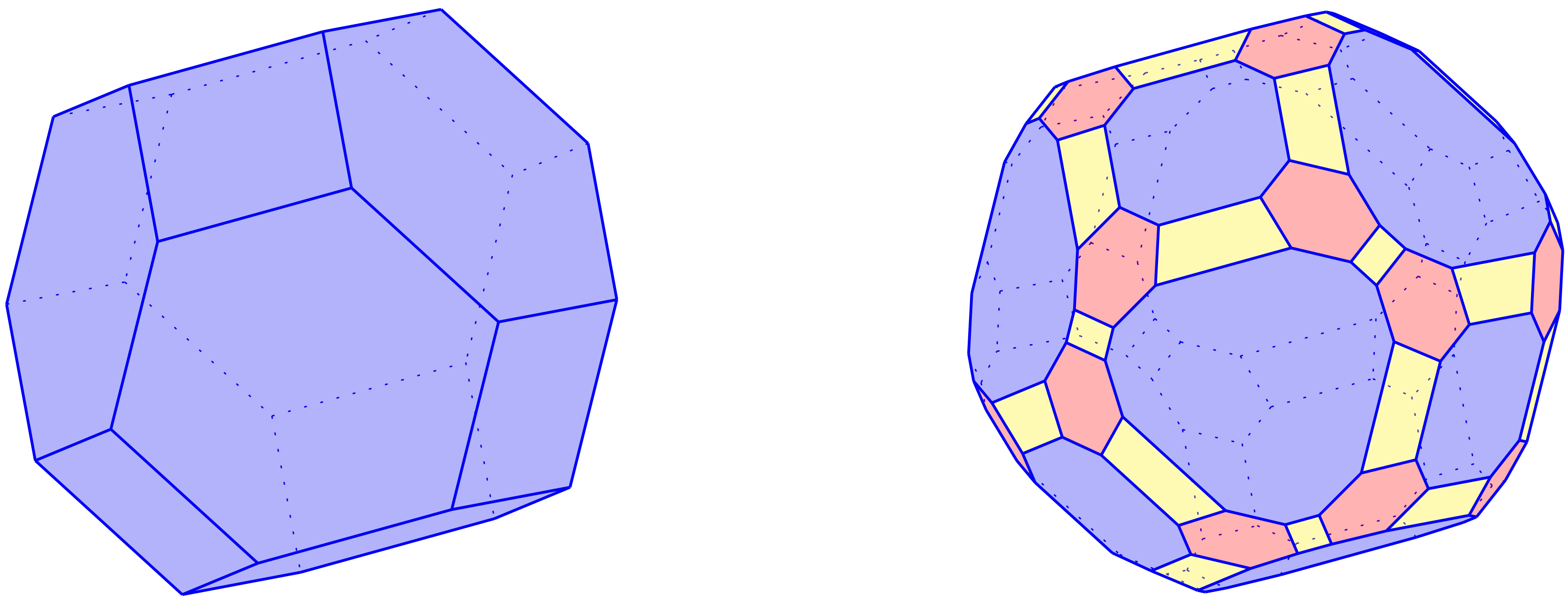

Figure 1: Π_3 and $\Pi_3^2(4, 1)$

Usual nested permutohedra

Definition (Formal)**.** Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \mathbb{R}^{d+1}$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{R}^d$ be two strictly increasing sequences such that entries in $\boldsymbol{\alpha}$ is sufficinetly larger than entries in $\boldsymbol{\beta}$.

For any $(\pi, \tau) \in \mathfrak{S}_{d+1} \times \mathfrak{S}_d$, we define

$$v_{\pi,\tau}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} := \underbrace{\sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_{\pi^{-1}(i)}}_{v_\pi^{\boldsymbol{\alpha}}} + \sum_{i=1}^{d} \beta_i \boldsymbol{f}^{\pi}_{\tau^{-1}(i)},$$

where for any permutation $\pi \in \mathfrak{S}_{d+1}$,

$$\boldsymbol{f}_i^{\pi} := \boldsymbol{e}_{\pi^{-1}(i+1)} - \boldsymbol{e}_{\pi^{-1}(i)}, \quad \forall 1 \le i \le d.$$

Then

$$\mathrm{Perm}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathrm{conv}\left(v_{\pi,\tau}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} \ : \ (\pi, \tau) \in \mathfrak{S}_{d+1} \times \mathfrak{S}_d \right).$$

Nested Braid fan

Fact. Br_d has $(d+1)!$ maximal cones, each of which is determined by *ordering of coordinates* associated with a permutation $\pi \in \mathfrak{S}_{d+1}$:

$$\sigma(\pi) := \{\boldsymbol{x} \in V^* \ : \ x_{\pi^{-1}(1)} \le x_{\pi^{-1}(2)} \le \cdots \le x_{\pi^{-1}(d+1)}\}.$$

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Definition. For each $\sigma(\pi)$ in Br_d, we subdivided it into $d!$ cones by considering *first differences of coordinates* associated with a permutation $\tau \in \mathfrak{S}_d$:

$$\sigma(\pi,\tau) := \left\{ \boldsymbol{x} \in V^* : \begin{array}{c} \underbrace{x_{\pi^{-1}(1)} \le \overbrace{x_{\pi^{-1}(2)}}_{\Delta_1} \le x_{\pi^{-1}(3)}}^{\Delta_2} \le \cdots \le \underbrace{x_{\pi^{-1}(d)} \le x_{\pi^{-1}(d+1)}}_{\Delta_d} \\ \Delta_{\tau^{-1}(1)} \le \Delta_{\tau^{-1}(2)} \le \cdots \le \Delta_{\tau^{-1}(d)} \end{array} \right\}.$$

The collection of cones $\sigma(\pi,\tau)$, together with all of their faces, forms the *nested Braid fan*, denoted by Br_d^2.

A connection

Recall that the Braid fan Br_d is the *normal fan* of any usual permutohedron $\mathrm{Perm}(\boldsymbol{\alpha})$.

Proposition (Castillo-L.)**.** *The nested Braid fan* Br_d^2 *is the* normal fan *of any usual nested permutohedron* $\mathrm{Perm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Generalized nested permutohedra

As a consequence of this connection, one can give two different but equivalent definitions for *generalized nested permutohedra*.

Definition. A *generalized nested permutohedron* is a polytope obtained from a usual nested permutohedron by moving the facets **without passing vertices**.

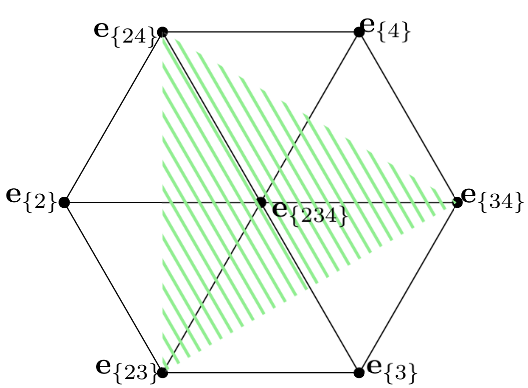
Definition. A polytope $P \in \mathbb{R}^{d+1}$ is a *generalized nested permutoheron* if its normal fan is *refined* by the nested Braid fan Br_d^2.

Example

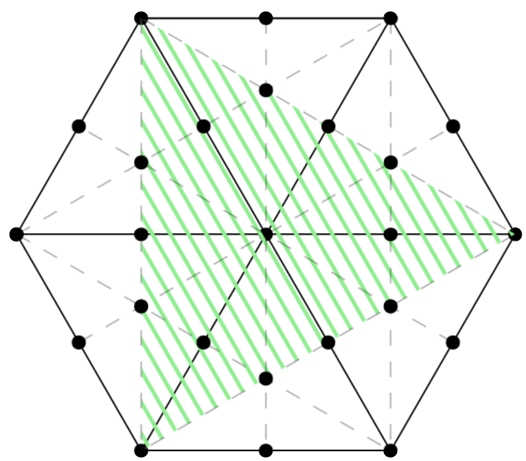
Recall our nonexample

$$Q = \text{conv}(\text{Perm}((1, 3, 3, 3)) \cup \text{Perm}(2, 2, 2, 4))).$$

Let σ be the normal cone of Q at the vertex $(1, 3, 3, 3)$. We showed that it *not* a coarsening cones in Br_3. However, it is a coarsening of cones in Br_3^2.



(a) σ in the Braid fan Br_3



(b) σ in the nested Braid fan Br_3^2

Results on generalized permutohedra and Braid fan

i. Explicity descriptions for usual permutohedra $\mathrm{Perm}(\boldsymbol{\alpha})$:

- Convex hull of $(d+1)!$ permutations of $\boldsymbol{\alpha}$.
- ○ Inequality description.

ii. Two different but equivalent defintions for generalized permutohedra:

- Moving facets of usual permutohedra.
- Normal fan coarsens the Braid fan.

iii. Nice combinatorics of the Braid fan Br_d :

- ○ k-dimensional cones in Br_d are in bijection with k-chains in the truncated Boolean algebra $\overline{\mathcal{B}A_{d+1}}$.

iv. Deformation cones of Braid fan.

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Inequality description of usual permutohedra

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Notation. For any $S \subseteq [d+1] = \{1, 2, \dots, d+1\}$, we define $\boldsymbol{e}_S = \sum_{i \in S} \boldsymbol{e}_i$.

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Theorem (Rado)**.** *The inequality description of the usual permutohedron* $\mathrm{Perm}(\alpha)$ *is given by*

$$\langle \boldsymbol{e}_{[d+1]}, \boldsymbol{x} \rangle = x_1 + \cdots + x_{d+1} \quad = \quad \alpha_1 + \cdots + \alpha_{d+1}$$

$$\langle \boldsymbol{e}_S, \boldsymbol{x} \rangle = \sum_{i \in S} x_i \quad \leq \quad \sum_{i=d+2-|S|}^{d+1} \alpha_i, \qquad \forall \emptyset \neq S \subsetneq [d+1].$$

Further, all those $2^{d+1} - 2$ *inequalities are facet defining.*

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Example. Let $\boldsymbol{\alpha} = (1, 2, 3)$. Then $\mathrm{Perm}(\boldsymbol{\alpha}) = \Pi_2$ is defined by the following linear system:

$$\langle \boldsymbol{e}_{[3]}, x \rangle = x_1 + x_2 + x_3 = 1 + 2 + 3 = 6,$$

$$\langle \boldsymbol{e}_1, \boldsymbol{x} \rangle = x_1 \le 3, \quad \langle \boldsymbol{e}_2, \boldsymbol{x} \rangle = x_2 \le 3, \quad \langle \boldsymbol{e}_3, \boldsymbol{x} \rangle = x_3 \le 3,$$

$$\langle \boldsymbol{e}_{\{1,2\}}, \boldsymbol{x} \rangle = x_1 + x_2 \le 5, \quad \langle \boldsymbol{e}_{\{1,3\}}, \boldsymbol{x} \rangle = x_1 + x_3 \le 5, \quad \langle \boldsymbol{e}_{\{2,3\}}, \boldsymbol{x} \rangle = x_2 + x_3 \le 5.$$

Further, these 6 inequalities all define facets of Π_2.

Results on generalized permutohedra and Braid fan

i. Explicity descriptions for usual permutohedra $\mathrm{Perm}(\boldsymbol{\alpha})$:

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- ○ Submodular Theorem.

Consequence of Rado's Theorem

(i) Any generalized permutohedra on $\mathbb{R}^{d+1}$ can be defined by the linear system:

$$\begin{aligned} \langle \boldsymbol{e}_{[d+1]}, \boldsymbol{x} \rangle &= b_{[d+1]} \\ \langle \boldsymbol{e}_S, \boldsymbol{x} \rangle &\le b_S, \qquad \forall \emptyset \neq S \subsetneq [d+1], \end{aligned}$$

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Each generalized permutohedron in $\mathbb{R}^{d+1}$ determines a unique $\boldsymbol{b} \in \mathbb{R}^{2^{d+1}-1}$. We call the collection of these $\boldsymbol{b}$'s the *deformation cone* $\mathrm{Def}(\mathrm{Br}_d)$ of the Braid fan Br_d.

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Question 1. Can we determine $\mathrm{Def}(\mathrm{Br}_d)$?

Consequence of Rado's Theorem

(ii) Recall that the normal fan of usual permutohedron in $\mathbb{R}^{d+1}$ is Br_d.

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Thus, Br_d has $2^{d+1}-2$ rays, i.e., 1-dimensional cones, indexed by nonempty proper subsets of $[d+1]$:

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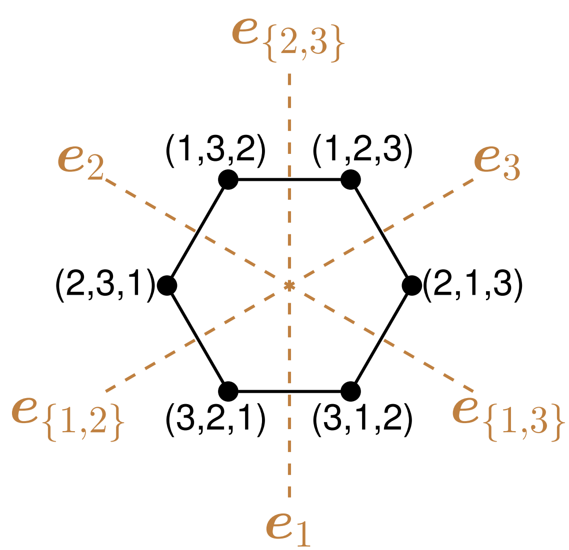

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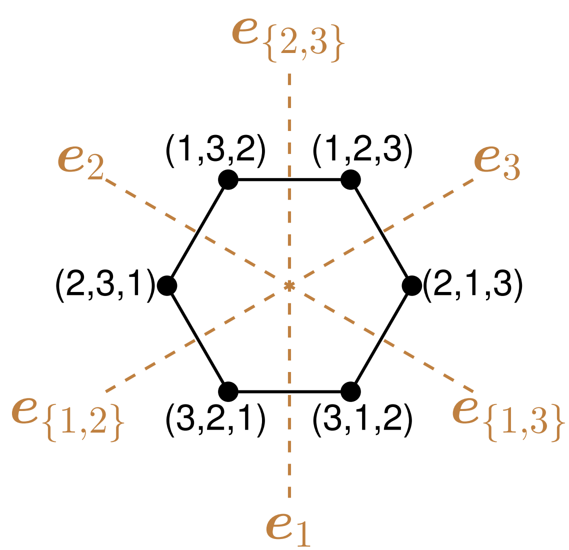

Question 2. What about other cones in Br_d?

Description of cones in Br_d

Recall that the *Boolean Algebra*, denoted by $\mathcal{B}A_{d+1}$, is the poset consisting of all subsets of $[d+1]$ ordered by containment. This poset has a minimal element $\hat{0} = \emptyset$ and a maximal element $\hat{1} = [d+1]$.

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Fact. The k-dimensional cones of Br_d are in bijection with k-chains in the *truncated Boolean algebra* $\overline{\mathcal{B}A_{d+1}}$, i.e. sequences of the form $\emptyset \neq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq [d+1]$.

Hence, the d-dimensional cones are in bijection with maximal chains in $\overline{\mathcal{B}A_{d+1}}$.

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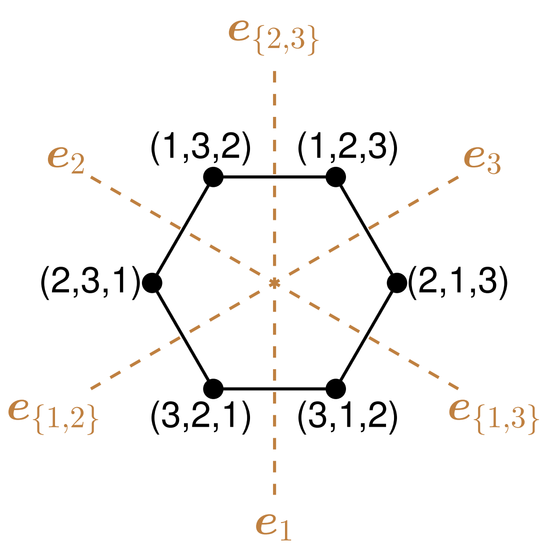

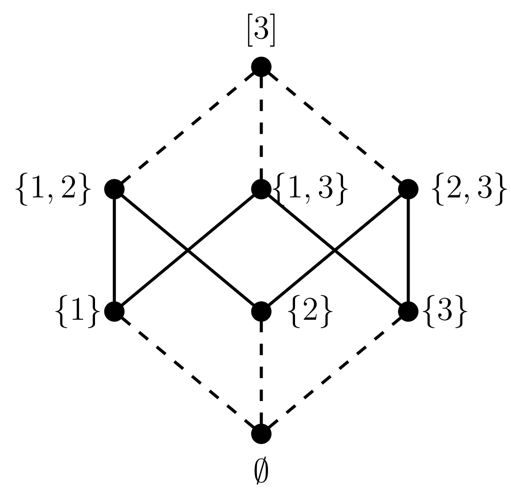

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Submodular Theorem

Theorem. *The deformation cone* $\mathrm{Def}(\mathrm{Br}_d)$ *is the set of* $\boldsymbol{b} \in \mathbb{R}^{2^{d+1}-1}$ *satisfying the following* submodular property*:*

$$b_{A\cap B} + b_{A\cup B} \leq b_A + b_B, \quad \forall A, B \subseteq [d+1],$$

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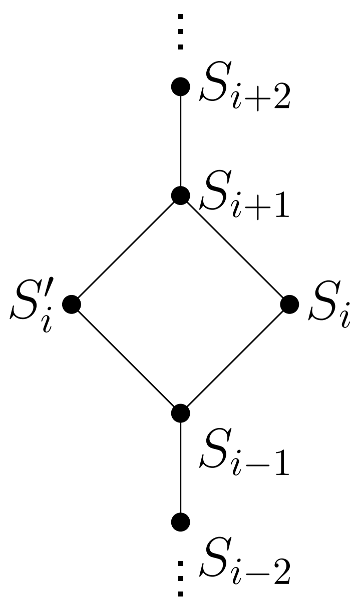
Remark 2. Another motivation for our work was to give a natural combinatorial proof for the Submodular Theorem.

Idea of the Proof

i. Each pair of adjacent maximal cones in Br_d provide an inequality to describe the deformation cone $\mathrm{Def}(\mathrm{Br}_d)$.

ii. Maximal cones are in bijection with maximal chains in $\overline{\mathcal{B}A_{d+1}}$.



$S_{i+1} = S_i \cup S_i'$ and $S_{i-1} = S_i \cap S_i'$

$$\Longrightarrow \quad \boldsymbol{e}_{S_{i+1}} + \boldsymbol{e}_{S_{i-1}} = \boldsymbol{e}_{S_i} + \boldsymbol{e}_{S_i'}$$

$$\Longrightarrow \quad z_{S_{i+1}} + z_{S_{i-1}} \leq z_{S_i} + z_{S_i'}$$

or equivalently $\quad z_{S_i \cup S_i'} + z_{S_i \cap S_i'} \leq z_{S_i'} + z_{S_i}$

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- Inequality description, where inequalities are indexed by nonempty proper subsets of $[d+1]$.

ii. Two different but equivalent defintions for generalized permutohedra:

- Moving facets of usual permutohedra.
- Normal fan coarsens the Braid fan.

iii. Nice combinatorics of the Braid fan Br_d :

- k-dimensional cones in Br_d are in bijection with k-chains in the truncated Boolean algebra $\overline{\mathcal{B}A_{d+1}}$.

iv. Deformation cones of Braid fan.

- Submodular Theorem.

Results on nested versions

i. Explicity descriptions for usual **nested** permutohedra $\mathrm{Perm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$:

- Convex hull of $(d+1)!d!$ points, corresponding to permutations of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.
- ○ Inequality description, where inequalities are indexed by ~~nonempty proper subsets~~ ordered set partitions of $[d+1]$.

ii. Two different but equivalent defintions for generalized **nested** permutohedra:

- Moving facets of usual **nested** permutohedra.
- Normal fan coarsens the **nested** Braid fan.

iii. Nice combinatorics of the **nested** Braid fan $\mathrm{Br}_d^{\mathbf{2}}$:

- ○ k-dimensional cones in $\mathrm{Br}_d^{\mathbf{2}}$ are in bijection with k-chains in the truncated ~~Boolean algebra $\overline{\mathcal{BA}_{d+1}}$~~ ordered set partition poset $\overline{\mathcal{O}_{d+1}}$.

iv. Deformation cones of **nested** Braid fan.

- ○ Submodular Theorem **+ ⅄ condition.**

THANK YOU!