On pure-cycle Hurwitz numbers

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Outline

- Describe the Hurwitz problem
- State our results: the pure-cycle case
- Connection to geometry

Hurwitz's problem

Definition 1. Given integers d and r, and r partitions $\lambda_1, \dots, \lambda_r \vdash d$, a *Hurwitz factorization* of $(d, r, (\lambda_1, \dots, \lambda_r))$ is an r-tuple $(\sigma_1, \dots, \sigma_r)$ satisfying the following conditions:

(i) $\sigma_i \in S_d$ has cycle type (or is in the conjugacy class) λ_i, for every i;

(ii) $\sigma_1 \cdots \sigma_r = 1$;

(iii) $M := \langle \sigma_1, \dots, \sigma_r \rangle$ is a transitive subgroup of S_d.

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Definition 2. We say two Hurwitz factorizations $(\sigma_1, \dots, \sigma_r)$ and $(\sigma_1', \dots, \sigma_r')$ are *equivalent* if they are related by simultaneous conjugation by an element of S_d, i.e., $\exists \tau \in S_d$ such that $\tau \sigma_i \tau^{-1} = \sigma_i'$.

We call the number of equivalent classes of Hurwitz factorizations of $(d, r, (\lambda_1, \dots, \lambda_r))$ the *Hurwitz number* $h(d, r, (\lambda_1, \dots, \lambda_r))$.

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Definition 2. We say two Hurwitz factorizations $(\sigma_1, \dots, \sigma_r)$ and $(\sigma'_1, \dots, \sigma'_r)$ are *equivalent* if they are related by simultaneous conjugation by an element of S_d, i.e., $\exists \tau \in S_d$ such that $\tau \sigma_i \tau^{-1} = \sigma'_i$.

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Question: What is the Hurwitz number $h(d, r, (\lambda_1, \dots, \lambda_r))$?

Braid group action

Let the *Artin braid group* B_r act on tuples $(\sigma_1, \ldots, \sigma_r)$ in S_d in the following way: the ith generator acts by replacing (σ_i, σ_{i+1}) by $(\sigma_{i+1}, \sigma_{i+1}^{-1}\sigma_i\sigma_{i+1})$. Note this action preserves the product of σ_i's and the group generated by σ_i's.

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By sending the ith generator to the transposition $(i, i+1)$, we obtain a natural map $B_r \longrightarrow S_r$. The kernel of this map is the *pure braid group*, which not only preserves $\sigma_1 \dots \sigma_r = 1$ and $M = \langle \sigma_1, \dots, \sigma_r \rangle$, but sends each σ_i to a conjugate of itself in M.

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Therefore, the pure braid group acts on the set of equivalent classes of Hurwitz factorizations.

Question: How many pure braid orbits are there?

The pure-cycle case

A number of people (Hurwitz, Goulden, Jackson, Vakil ...) have studied Hurwitz numbers. However, they restricted their attention to the case where all but one or two σ_i's are transpositions.

We consider instead the *pure-cycle* case. This means each λ_i has the form $(e_i, 1, \dots, 1)$, for some $e_i \geq 2$, or equivalently, each σ_i is an e_i cycle.

Our main theorem is in the *genus*-0 case, which simply means that

$$2d - 2 = \sum_{i=1}^{r} (e_i - 1).$$

Lemma 3 (L-Osserman). *In the genus-0 pure-cycle case, when $r = 3$,*

$$h(d, 3, (\lambda_1, \lambda_2, \lambda_3)) = 1.$$

Theorem 4 (L-Osserman). *In the genus-0 pure-cycle case, when $r = 4$,*

$$h(d, 4, (\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = \min\{e_i(d + 1 - e_i)\}$$

Theorem 5 (L-Osserman)**.** *In the genus-0 pure-cycle case, every equivalence class of Hurwitz factorizations of $(d, r, (\lambda_1, \dots, \lambda_r))$ is in a single pure braid orbit.*

Our original motivation for studying this situation was geometric, but it appears that there is also a lot of interesting combinatorial structure. The genus-0 pure-cycle case seems to become more complicated when $r > 4$. However, we have:

Conjecture 6. *In the genus-0 pure-cycle case,*

(i) if $\max e_i = d$, then $h(d, r, (\lambda_1, \dots, \lambda_r)) = d^{r-3}$.

(ii) if $\max e_i = d - 1$, then $h(d, r, (\lambda_1, \dots, \lambda_r)) = (r-2)(d-1)^{r-3}$.

We also have a result outside of the genus-0 case:

Lemma 7. *In the pure-cycle case and $r = 3$, and $2d = \sum_{i=1}^{3}(e_i - 1)$, the Hurwitz number is*

$$h(d, 3, (\lambda_1, \lambda_2, \lambda_3)) = \lceil \frac{\prod_{i=1}^{3} \binom{f_i}{2}}{3} \rceil,$$

where $f_i = (\sum_{j=1}^{3} e_j - 2e_i + 1)/2$.

Branched Covers

Hurwitz numbers count certain kinds of branched covers of Riemann surfaces.

Given C a compact Riemann surface, we consider *branched cover* (f, C') of C, where f is a surjective map from C' to C, and C' is a compact Riemann surface.

Properties of a branched cover:

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Properties of a branched cover:

- For any $p \in C'$, $\exists e \geq 1$, s.t., in a neighborhood of p, f acts like $z \to z^e$ in a neighborhood of 0. We define the *ramification index* of f at p to be $e(p) := e$. When $e > 1$, we call p a *ramification point*.

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- For any $q \in C$, if $\exists p \in f^{-1}(q)$, such that p is a ramification point, then we call q a *branch point*.
- Compactness $\Rightarrow$ finitely many ramification points $\Rightarrow$ finitely many branch points.
- $\exists d$ s.t. for any $q \in C$, $\sum_{p \in f^{-1}(q)} e(p) = d$. We call d the degree of this cover.

  (1) q not a branch point: $|f^{-1}(q)| = d$;

  (2) q a branch point: if $f^{-1}(q) = \{p_1, \ldots, p_k\}$, define $\lambda(q) := \{e(p_1), \ldots, e(p_k)\} \vdash d$ to be the branch type of q.

From now on, we fix $C = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \cong S^2$.

Example: Let $f : \mathbb{P}^1 \to \mathbb{P}^1, z \mapsto z^3$. $(f, \mathbb{P}^1)$ is a branched cover of $\mathbb{P}^1$ of degree 3. It has two branch points 0 and ∞. Their branch types are both (3).

Question: How many (connected) branched covers up to isomorphism are there of C of degree d, with r branched points $q_1, \ldots, q_r$ of branch types $\lambda_1, \ldots, \lambda_r \vdash d$?

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We consider $(f_1, C') \sim (f_2, C'')$ if there exists an isomorphism $i : C' \to C''$ s.t. $f_1 = f_2 \circ i$.

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NOTE: By the Riemann-Hurwitz formula and the fact that $g(C) = g(S^2) = 0$,

$$2d - 2 + 2g(C') = \sum_{q \text{ a branch point}} \sum_{p \in f^{-1}(q)} (e(p) - 1) = \sum_{i=1}^{r} \sum_{j} (\lambda_{i,j} - 1),$$

where $\lambda_{i,j}$ is the jth part of λ_i. The genus $g(C')$ of C' is determined when given $(d, r, (\lambda_1, \dots, \lambda_r))$.

Monodromy map

Let $X := \mathbb{P}^1 \setminus \{q_1, \dots, q_r\}$ and choose a base point q in X.

$\pi_1(X, q) = \langle \gamma_1, \dots, \gamma_r \rangle / (\gamma_1 \dots \gamma_r = 1)$, if we choose suitable γ_i as a loop from q to q around q_i.

Given any cover (f, C') of $\mathbb{P}^1$, construct a group homomorphism μ as follows:

q has d preimages; label them as $1, 2, \dots, d$.

Let $\mu : \pi_1(X, q) \to sym(f^{-1}(q)) \cong S_d$ be defined by $\mu(\gamma) = \sigma$, where $\sigma(i) = j$ if when we lift γ to C' starting from i, it ends at j.

μ is the *monodromy map*. We call $M := \mu(\pi_1(X, q))$ the *monodromy group* of the cover.

Let $\sigma_i = \mu(\gamma_i)$, then σ_i is a permutation of cycle type λ_i.

$(\gamma_1, \dots, \gamma_r) \mapsto (\sigma_1, \dots, \sigma_r)$ a **Hurwitz factorization** of $(d, r, (\lambda_1, \dots, \lambda_r))$.

Hurwitz numbers revisited

Theorem 8 (Riemann Existence Theorem)**.** *Given any equivalence class of Hurwitz factorizations* $(\sigma_1, \dots, \sigma_r)$ *of* $(d, r, (\lambda_1, \dots, \lambda_r))$, *there exists a connected branched cover* (f, C') *of* $\mathbb{P}^1$, *so that under the corresponding monodromy map, we get* $(\sigma_1, \dots, \sigma_r)$. *Also the genus* g *of* C' *satisfies:*

$$2d - 2 + 2g = \sum_{i=1}^{r} \sum_{j} (\lambda_{i,j} - 1).$$

Therefore, the **Hurwitz number** $h(d, r, (\lambda_1, \dots, \lambda_r))$ counts the number of connected branched covers up to isomorphism of $\mathbb{P}^1$ of degree d, with r branched points $q_1, \dots, q_r$ of branch types $\lambda_1, \dots, \lambda_r$.

Hurwitz space

The *Hurwitz space* $H(d, r, (\lambda_1, \ldots, \lambda_r))$ is the collection of $(q_1, \ldots, q_r, (f, C'))$ where (f, C') is a branched cover of $\mathbb{P}^1$ of degree d with r branched points $q_1, \ldots, q_r$ of branch types $\lambda_1, \ldots, \lambda_r$. We can put a natural topology on it.

$H(d, r, (\lambda_1, \ldots, \lambda_r))$ is a cover of $U^r = \{(q_1, \ldots, q_r) \mid q_i \neq q_j\} \subset (\mathbb{P}^1)^r$. The degree of this cover is the Hurwitz number $h(d, r, (\lambda_1, \ldots, \lambda_r))$.

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Hence, we can restate our earlier theorem:

Theorem 9 (L-Osserman)**.** *If each λ_i has the form $(e_i, 1, \ldots, 1)$, and $2d - 2 = \sum_{i=1}^r (e_i - 1)$, then the Hurwitz space $H(d, r, (\lambda_1, \ldots, \lambda_r))$ is connected.*