Perturbation of transportation polytopes

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Workshop on Convex Polytopes

Kyoto, Japan

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Outline

- Multivariate generating function
- A perturbation method
- Transportation polytopes
  - Introduction to transportation polytope
  - Multivariate generating functions of transportation polytopes

PART I:

Multivariate generating function

***Summary*: We will go over basic definitions and theory related to the multivariate generating functions.**

Polyhedra and polytopes

Definition 1 (Normal Specification)**.** A *polyhedron* $P \subset \mathbb{R}^D$ is the solution set of a (finite) system of linear inequalities:

$$P = \{\mathbf{x} \in \mathbb{R}^D : A\mathbf{x} \leq \mathbf{b}\},$$

for some $A \in \mathbb{R}^{N \times D}, \mathbf{b} \in \mathbb{R}^N$.

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Lemma 2 (Canonical Specification)**.** *A polyhedron is pointed if and only if it is affinely equivalent to a polyhedron defined as*

$$P = \{\mathbf{x} \in \mathbb{R}^{D'} : A'\mathbf{x} = \mathbf{b}', \mathbf{x} \geq 0\},$$

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An *integral* polytope/polyhedron is a polytope/polyhedron whose vertices are all lattice points. i.e., points with integer coordinates.

Multivariate generating function

For any polyhedron $P \in \mathbb{R}^D$, we define the *multivariate generating function* (MGF) of P as

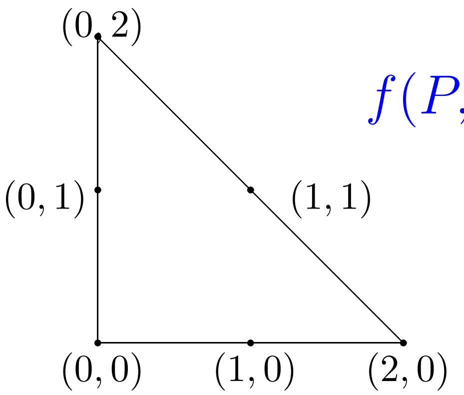
$$f(P, \mathbf{z}) = \sum_{\alpha \in P \cap \mathbb{Z}^D} \mathbf{z}^\alpha,$$

where $\mathbf{z}^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_D^{\alpha_d}$.

One sees that by setting $\mathbf{z} = (1, 1, \dots, 1)$, we get the number of lattice points in P if P is a polytope.

Example: Let P be the polytope with vertices $v_1 = (0, 0)$, $v_2 = (2, 0)$ and $v_3 = (0, 2)$.

P :



$$\begin{aligned} f(P, \mathbf{z}) &= z_1^0 z_2^0 + z_1^1 z_2^0 + z_1^2 z_2^0 + z_1^0 z_2^1 + z_1^1 z_2^1 + z_1^0 z_2^2 \\ &= 1 + z_1 + z_1^2 + z_2 + z_1 z_2 + z_2^2. \end{aligned}$$

Brion's Lemma

Definition 3. Suppose v is a vertex of P. The feasible cone of P at v is:

$$\mathrm{fcone}(P, v) = \{u \in \mathbb{R}^d \ : \ v + \delta u \in P \text{ for all sufficiently small } \delta > 0\}.$$

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Lemma 4 (Brion, 1988; Lawrence, 1991)**.** *Let P be an integral polyhedron and let $V(P)$ be the vertex set of P. Then, considered as rational funtions,*

$$f(P, \mathbf{z}) = \sum_{v \in V(P)} \mathbf{z}^v f(\mathrm{fcone}(P, v), \mathbf{z}).$$

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Remark 5*.* Brion's lemma was stated more generally in terms of rational polyhedra. However, we only need the version for integral polyhedra.

Unimodular cones

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Lemma 6. *Suppose K is a d-dimensional cone in $\mathbb{R}^D$, generated by vectors $\{r_i\}_{1 \le i \le d}$ such that the r_i's form a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^D \cap \mathrm{span}(\{r_i\})$. We call such a cone a unimodular cone. Then we have*

$$f(K, \mathbf{z}) = \prod_{i=1}^{d} \frac{1}{1 - \mathbf{z}^{r_i}}.$$

An example of the lemmas

Example: Let P be the polytope with vertices $v_1 = (0,0)$, $v_2 = (2,0)$ and $v_3 = (0,2)$.

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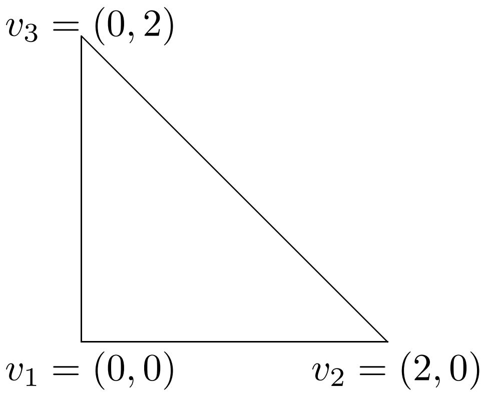

$f(P,\mathbf{z}) = \sum_{i=1}^{3} \mathbf{z}^{v_i} f(\mathrm{fcone}(P,v_i),\mathbf{z})$ (By Brion's lemma)

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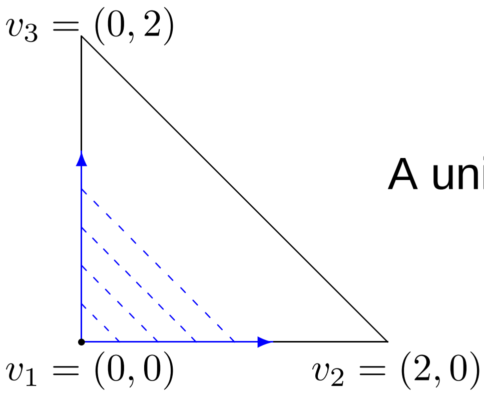

A unimodular cone generated by vectors $(1,0)$ and $(0,1)$.

$$f(P,\mathbf{z}) = \sum_{i=1}^{3} \mathbf{z}^{v_i} f(\text{fcone}(P,v_i),\mathbf{z})$$

$$= \mathbf{z}^{(0,0)} \frac{1}{(1-\mathbf{z}^{(1,0)})(1-\mathbf{z}^{(0,1)})}$$

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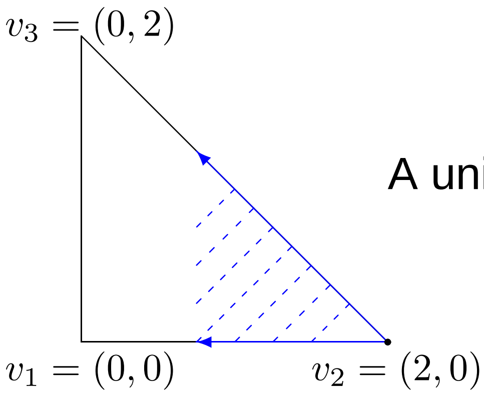

A unimodular cone generated by vectors $(-1,0)$ and $(-1,1)$.

$$f(P,\mathbf{z}) = \sum_{i=1}^{3} \mathbf{z}^{v_i} f(\mathrm{fcone}(P, v_i), \mathbf{z})$$

$$= \mathbf{z}^{(0,0)} \frac{1}{(1-\mathbf{z}^{(1,0)})(1-\mathbf{z}^{(0,1)})} + \mathbf{z}^{(2,0)} \frac{1}{(1-\mathbf{z}^{(-1,0)})(1-\mathbf{z}^{(-1,1)})}$$

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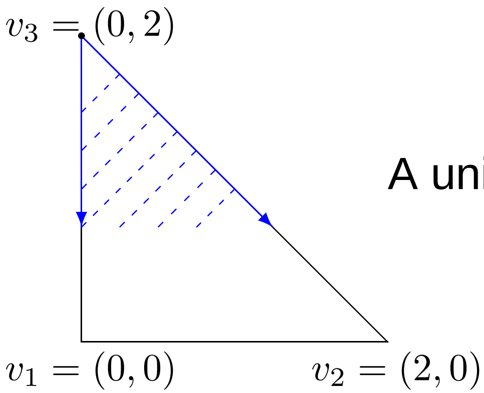

A unimodular cone generated by vectors $(0,-1)$ and $(1,-1)$.

$$f(P,\mathbf{z}) = \sum_{i=1}^{3} \mathbf{z}^{v_i} f(\mathrm{fcone}(P,v_i),\mathbf{z})$$

$$= \mathbf{z}^{(0,0)} \frac{1}{(1-\mathbf{z}^{(1,0)})(1-\mathbf{z}^{(0,1)})} + \mathbf{z}^{(2,0)} \frac{1}{(1-\mathbf{z}^{(-1,0)})(1-\mathbf{z}^{(-1,1)})}$$

$$+ \mathbf{z}^{(0,2)} \frac{1}{(1-\mathbf{z}^{(0,-1)})(1-\mathbf{z}^{(1,-1)})}$$

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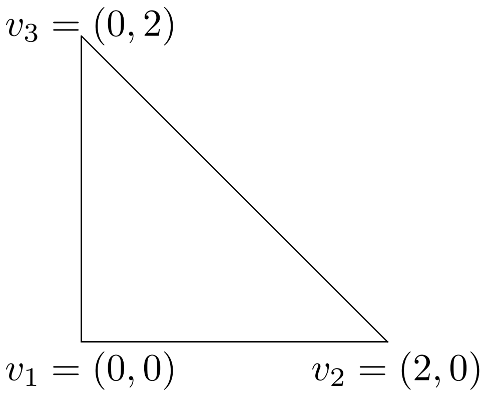

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$$+ \mathbf{z}^{(0,2)} \frac{1}{(1-\mathbf{z}^{(0,-1)})(1-\mathbf{z}^{(1,-1)})}$$

$$= \frac{1}{(1-z_1)(1-z_2)} + \frac{z_1^2}{(1-z_1^{-1})(1-z_1^{-1}z_2)} + \frac{z_2^2}{(1-z_2^{-1})(1-z_1 z_2^{-1})}$$

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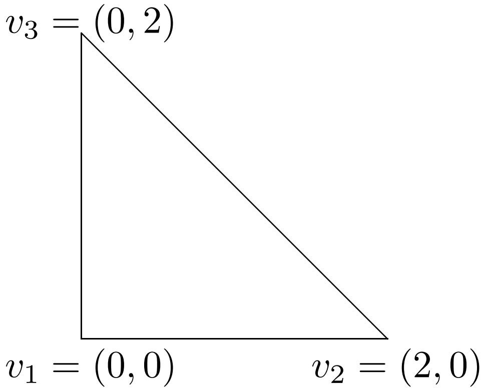


$$
\begin{aligned}
f(P,\mathbf{z}) &= \sum_{i=1}^{3} \mathbf{z}^{v_i} f(\mathrm{fcone}(P,v_i),\mathbf{z}) \\
&= \mathbf{z}^{(0,0)} \frac{1}{(1-\mathbf{z}^{(1,0)})(1-\mathbf{z}^{(0,1)})} \quad + \quad \mathbf{z}^{(2,0)} \frac{1}{(1-\mathbf{z}^{(-1,0)})(1-\mathbf{z}^{(-1,1)})} \\
&\qquad + \quad \mathbf{z}^{(0,2)} \frac{1}{(1-\mathbf{z}^{(0,-1)})(1-\mathbf{z}^{(1,-1)})} \\
&= \frac{1}{(1-z_1)(1-z_2)} + \frac{z_1^2}{(1-z_1^{-1})(1-z_1^{-1}z_2)} + \frac{z_2^2}{(1-z_2^{-1})(1-z_1 z_2^{-1})} \\
&= 1 + z_1 + z_1^2 + z_2 + z_1 z_2 + z_2^2
\end{aligned}
$$

PART II:

A perturbation method

***Summary*:** We introduce a perturbation method that reduces the problem of finding the MGF of a non-simple polytope to computing the MGFs of simple polytopes.

A perturbation method

When we calculate the MGF of a non-simple cone, the common method involves triangulating the cone into simple cones. We replace the triangulation step with a perturbation method.

Theorem 7 (L.). *Suppose* $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ *is a non-empty integral polyhedron in* $\mathbb{R}^D$ *and* $\mathbf{b}(t)$ *is a continuous function on some interval containing* 0 *satisfying the following conditions.*

a) $\mathbf{b}(t) \to \mathbf{b}$ *as* $t \to 0$.

b) For each $t \neq 0$ *in the interval,* $P(t) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}(t)\}$ *is a non-empty polyhedron with exactly* ℓ *vertices:* $w_{t,1}, \dots, w_{t,\ell}$*, and the feasible cone of* $P(t)$ *at* $w_{t,j}$ *does not depend on* t*, that is, for each* $j : 1 \leq j \leq \ell$*, there exists a fixed cone* K_j *such that* $\mathrm{fcone}(P(t), w_{t,j}) = K_j$ *for all* $t \neq 0$.

Then

$$f(P, \mathbf{z}) = \sum_{j=1}^{\ell} \mathbf{z}^{\lim_{t\to 0} w_{t,j}} f(K_j, \mathbf{z}).$$

An example of Theorem 7

Let

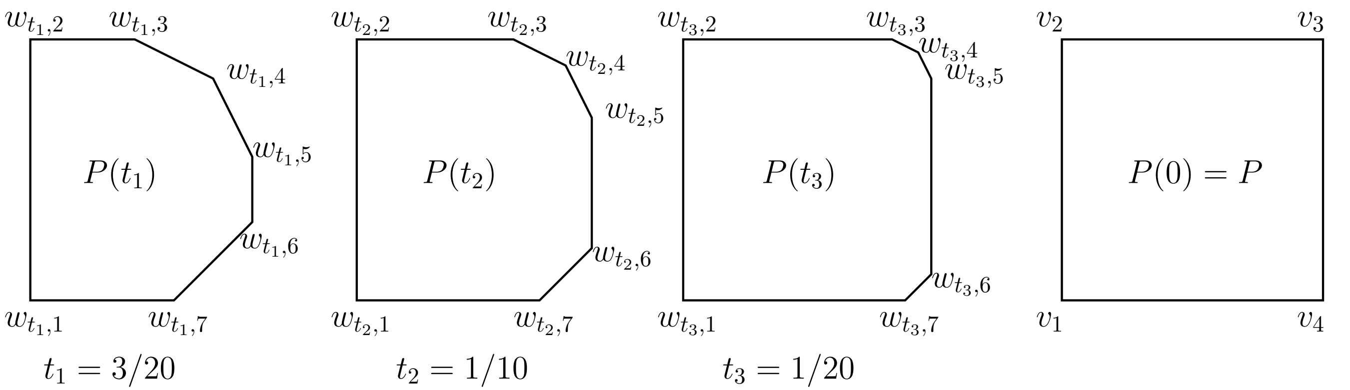
$$A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & -1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 3 \\ 3 \end{pmatrix}, \ \text{and } \mathbf{b}(t) = \begin{pmatrix} 1 \\ t \\ 1 \\ 0 \\ 1-2t \\ 3-3t \\ 3-3t \end{pmatrix}, 0 \leq t < 1/5.$$

Then $P = \{\mathbf{x} \in \mathbb{R}^2 \mid A\mathbf{x} \leq \mathbf{b}\}$ is just the unit square with vertices

$$v_1 = (0,0), v_2 = (0,1), v_3 = (1,1) \text{ and } v_4 = (1,0).$$

Condition a): $\mathbf{b}(t) \to \mathbf{b}$ **as** $t \to 0$.

An example of Theorem 7 (cont'd)



For any $t \in (0, 1/5)$, the polygon $P(t) = \{\mathbf{x} | A\mathbf{x} \leq \mathbf{b}(t)\}$ has seven vertices

$$w_{t,1} = (t, 0), \quad w_{t,2} = (t, 1),$$
$$w_{t,3} = (1 - 3t, 1), \quad w_{t,4} = (1 - t, 1 - t), \quad w_{t,5} = (1, 1 - 3t),$$
$$w_{t,6} = (1, 2t), \quad w_{t,7} = (1 - 2t, 0).$$

$$w_{t,1} \to v_1, \quad w_{t,2} \to v_2, \quad w_{t,3}, w_{t,4}, w_{t,5} \to v_3, \quad w_{t,6}, w_{t,7} \to v_4.$$

Condition b) The feasible cone of $P(t)$ at $w_{t,j}$ does not depend on t (for $t \in (0, 1/5)$). Let $K_j := \mathrm{fcone}(P(t), w_{t,j})$. By Theorem 7, we have

$$\begin{aligned} f(P, z) &= \mathbf{z}^{v_1} f(K_1, \mathbf{z}) + \mathbf{z}^{v_2} f(K_2, \mathbf{z}) + \mathbf{z}^{v_3} (f(K_3, \mathbf{z}) + f(K_4, \mathbf{z}) + f(K_5, \mathbf{z})) \\ &\quad + \mathbf{z}^{v_4} (f(K_6, \mathbf{z}) + f(K_7, \mathbf{z})). \end{aligned}$$

Total unimodularity

Our perturbation method is most efficient when the defining matrix is *totally unimodular*.

A *totally unimodular matrix* is a matrix whose minors are $0, 1$ or -1.

Lemma 8. *Suppose A is a totally unimodular matrix, and P is defined by the canonical specification*

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0. \tag{9}$$

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Recall that the MGF of a unimodular cone K is easy to calculate:

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Suppose A is a totally unimodular matrix, and P is an integral polytope defined by the canonical specification (9). We have the following two situations.

Total unimodularity (cont'd)

(i) If P is simple, the MGF of P is given by:

$$f(P,\mathbf{z}) = \sum_{v \in V(P)} \mathbf{z}^v f(\mathrm{fcone}(P,v),\mathbf{z}) = \sum_{v \in V(P)} \mathbf{z}^v \prod \frac{1}{1-\mathbf{z}^{r_{v,i}}},$$

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(ii) Suppose P is not simple and $\mathbf{b}(t)$ is a continuous function such that the perturbed polytopes

$$P(t) = \{\mathbf{x} \,:\, A\mathbf{x} = \mathbf{b}(t), \mathbf{x} \geq 0\}$$

are simple and satisfy the conditions of our perturbation theorem. Then

$$\begin{aligned} f(P,\mathbf{z}) &= \sum_{j=1}^{\ell} \mathbf{z}^{\lim_{t\to 0} w_{t,j}} f(K_j,\mathbf{z}) \\ &= \sum_{j=1}^{\ell} \mathbf{z}^{\lim_{t\to 0} w_{t,j}} \prod \frac{1}{1-\mathbf{z}^{r_{i,j}}} \end{aligned}$$

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A continuous function $\mathbf{b}(t)$ satisfying the above conditions exists: Choose $\mathbf{b}(t) = \mathbf{b} + t\mathbf{v}$, where $\mathbf{v}$ is a generic vector.

Total unimodularity (cont'd)

Suppose A is totally unimodular. What do we need to know to figure out the MGFs of (integral) polytopes defined by the canonical specification

$$P_{\mathbf{b}} := \{\mathbf{x} \,:\, A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}?$$

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(i) When is the polytope $P_{\mathbf{b}}$ *integral*?

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(iii) Fixing $\mathbf{b}$, how to find the *vertices* of $P_{\mathbf{b}}$?

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(iii) Fixing $\mathbf{b}$, how to find the *vertices* of $P_{\mathbf{b}}$?

(iv) Fixing $\mathbf{b}$ and a vertex v of $P_{\mathbf{b}}$, how to find the *generating rays of the feasible cone* of $P_{\mathbf{b}}$ at v?

PART III:

Transportation polytopes

***Summary*:** We discuss known results on transportation polytopes that are related to our talk, and then give formulas for MGFs of transportation polytopes by using our perturbation method.

Basic definitions

Suppose $\mathbf{r} = (r_1, \dots, r_m)$ and $\mathbf{c} = (c_1, \dots, c_n)$ two vectors of positive entries whose coordinates sum to a fixed number. The *transportation polytope* $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is the set of all $m \times n$ nonnegative matrices in which row i has sum r_i and column j has sum c_j. (In statistics, those matrices are called *contingency table with margins* $\mathbf{r}$ *and* $\mathbf{c}$.) We call $\mathcal{T}(\mathbf{r}, \mathbf{c})$ a *transportation polytope of order* $m \times n$.

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If $r_1 = r_2 = \cdots = r_m$ and $c_1 = c_2 = \cdots = c_n$, we say $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is a *central* transportation polytope of order $m \times n$.

Basic definitions

Suppose $\mathbf{r} = (r_1, \ldots, r_m)$ and $\mathbf{c} = (c_1, \ldots, c_n)$ two vectors of positive entries whose coordinates sum to a fixed number. The *transportation polytope* $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is the set of all $m \times n$ nonnegative matrices in which row i has sum r_i and column j has sum c_j. (In statistics, those matrices are called *contingency table with margins* $\mathbf{r}$ *and* $\mathbf{c}$.) We call $\mathcal{T}(\mathbf{r}, \mathbf{c})$ a *transportation polytope of order* $m \times n$.

If $r_1 = r_2 = \cdots = r_m$ and $c_1 = c_2 = \cdots = c_n$, we say $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is a *central* transportation polytope of order $m \times n$.

Example 10. Let $\mathbf{r} = (3, 3)$ and $\mathbf{c} = (2, 2, 2)$. Then $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is a central transportation polytope of order 2×3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \in \mathcal{T}(\mathbf{r}, \mathbf{c}).$$

Canonical specification

It is easy to give a canonical specification of a transportation polytope.

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$$\left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix} : \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} \\ x_{2,1} \\ x_{1,2} \\ x_{2,2} \\ x_{1,3} \\ x_{2,3} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad x_{i,j} \geq 0 \right\},$$

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where the above matrix is denoted by $A_{2,3}$.

$\mathcal{T}(\mathbf{r}, \mathbf{c})$ has a natural canonical specification:

$$\mathcal{T}(\mathbf{r}, \mathbf{c}) = \left\{ \mathbf{x} \, : \, A_{m,n}\mathbf{x} = \begin{pmatrix} \mathbf{r}^T \\ \mathbf{c}^T \end{pmatrix}, \quad \mathbf{x} \geq 0 \right\},$$

where $A_{m,n}$ is the $(m+n) \times mn$ incidence matrix of $K_{m,n}$.

Questions

It turns out that $A_{m,n}$ is totally unimodular. Recall we need to know the followings to figure out the MGFs of integral transportation polytopes $\mathcal{T}(\mathbf{r}, \mathbf{c})$ of order $m \times n$.

(i) When is the polytope $\mathcal{T}(\mathbf{r}, \mathbf{c})$ *integral*?

(ii) When is the polytope $\mathcal{T}(\mathbf{r}, \mathbf{c})$ *simple*?

(iii) How to find the *vertices* of $\mathcal{T}(\mathbf{r}, \mathbf{c})$?

(iv) How to find the *generating rays of the feasible cone* of $\mathcal{T}(\mathbf{r}, \mathbf{c})$ at a given vertex?

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(iv) How to find the *generating rays of the feasible cone* of $\mathcal{T}(\mathbf{r}, \mathbf{c})$ at a given vertex? **Answer**: Also using auxiliary graphs; but we will skip details.

Non-degeneracy

Definition 12. The transportation polytope $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is *non-degenerate* if the only nonempty index subsets $I \subseteq [m]$ and $J \subseteq [n]$ satisfying $\sum_{i \in I} r_i = \sum_{j \in J} c_j$ are $I = [m]$ and $J = [n]$; otherwise it is *degenerate*.

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Example 13. $\mathcal{T}((2,3,6,6),(1,4,5,7))$ is degenerate because $2 + 3 = 5$ (or $3 + 6 = 4 + 5$, or . . .)

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Fact. Every non-degenerate transportation polytope is simple.

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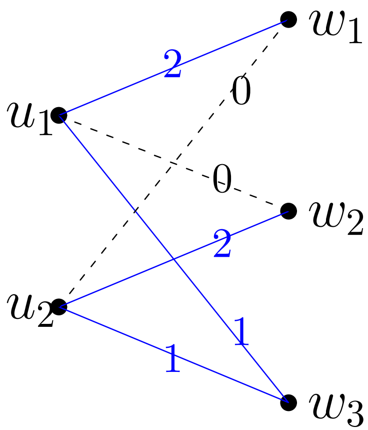

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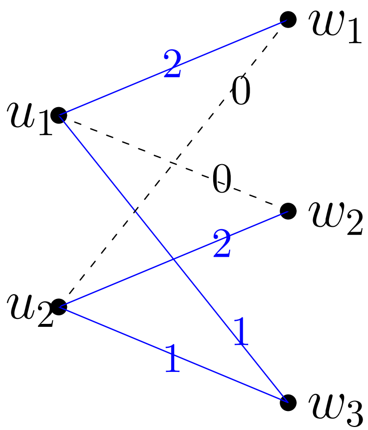

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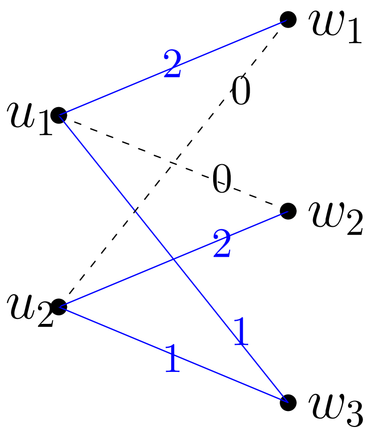
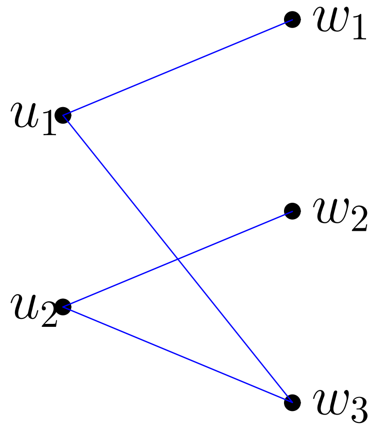


$\longrightarrow$



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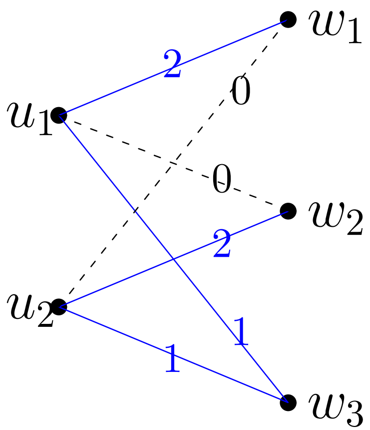
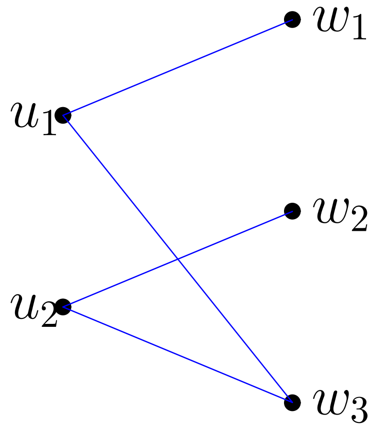


$$\xrightarrow{\text{aux}}$$



$M \in \mathcal{T}(\mathbf{r}, \mathbf{c})$

subgraph of $K_{m,n}$ with a positive labelling *with margin* $(\mathbf{r}, \mathbf{c})$

auxiliary Graph $\mathrm{aux}(M)$

Vertices

Theorem 14. *Suppose $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is non-degenerate and $M \in \mathcal{T}(\mathbf{r}, \mathbf{c})$. Then M is a vertex of $\mathcal{T}(\mathbf{r}, \mathbf{c})$ if and only if $\mathrm{aux}(M)$ is a spanning tree of $K_{m,n}$.*

Furthermore, no two vertices of $\mathcal{T}(\mathbf{r}, \mathbf{c})$ have the same auxiliary graphs.

Hence, when $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is non-degenerate, the map aux induces a bijection:

$$\left\{\text{vertices of } \mathcal{T}(\mathbf{r}, \mathbf{c})\right\} \longleftrightarrow \left\{\begin{array}{c}\text{spanning trees of } K_{m,n} \\ \text{that admit positive labellings} \\ \text{with margin } (\mathbf{r}, \mathbf{c})\end{array}\right\}$$

MGFs of non-degenerate integral transportation polytopes

Corollary 15. *Suppose $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is a non-degenerate integral transportation polytope. Then*

$$f(\mathcal{T}(\mathbf{r}, \mathbf{c}), \mathbf{z}) = \sum_M \mathbf{z}^M \prod \frac{1}{1 - \mathbf{z}^{r_{M,i}}}$$

where the summation is over all vertices M of $\mathcal{T}(\mathbf{r}, \mathbf{c})$, and $\{r_{M,i}\}$ is the set of vectors that generates the unimodular cone $\mathrm{fcone}(\mathcal{T}(\mathbf{r}, \mathbf{c}), M)$.

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where the summation is over all spanning trees T of $K_{m,n}$ that admits a positive labelling with margin $(\mathbf{r}, \mathbf{c})$, and M_T is the vertex corresponding to T, and $\{r_{M_T,i}\}$ is the set of vectors that generates the unimodular cone $\mathrm{fcone}(\mathcal{T}(\mathbf{r}, \mathbf{c}), M_T)$.

MGFs of non-degenerate integral transportation polytopes

Corollary 15. *Suppose* $\mathcal{T}(\mathbf{r}, \mathbf{c})$ *is a non-degenerate integral transportation polytope. Then*

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where the summation is over all spanning trees T *of* $K_{m,n}$ *that admits a positive labelling with margin* $(\mathbf{r}, \mathbf{c})$*, and* M_T *is the vertex corresponding to* T*, and* $\mathrm{cycle}(T, e)$ *is an* $m \times n$ $(0, 1, -1)$*-matrix associated to the unique cycle in* $T \cup e$*.*

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Question: What if $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is degenerate?

Answer: We construct a perturbation that works for any transportation polytope, which gives a formula similar to the above formula.

Perturbation of (degenerate) transportation polytopes

Suppose $\mathbf{r} = (r_1, r_2, \ldots, r_m)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ are two rational positive vectors. We define

$$\mathbf{r}(t) = (r_1 - t, \ldots, r_m - t), \ \mathbf{c}(t) = (c_1, \ldots, c_{n-1}, c_n - mt).$$

We show that $\{\mathcal{T}(\mathbf{r}(t), \mathbf{c}(t)) \mid t \in (0, \epsilon)\}$ is a family of non-degenerate transportation polytopes satisfying the conditions of our perturbation theorem, where ϵ is a sufficiently small positive number.

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Corollary 16. *Suppose $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is an integral transportation polytope. Then*

$$f(\mathcal{T}(\mathbf{r}, \mathbf{c}), \mathbf{z}) = \sum_T \mathbf{z}^{M_T} \prod_{e \notin E(T)} \frac{1}{1 - \mathbf{z}^{\mathrm{cycle}(T,e)}}, \tag{17}$$

where the summation is over all spanning trees of $K_{m,n}$ that admits a positive labelling with margin $(\mathbf{r}(t), \mathbf{c}(t))$ (for sufficiently small ϵ).

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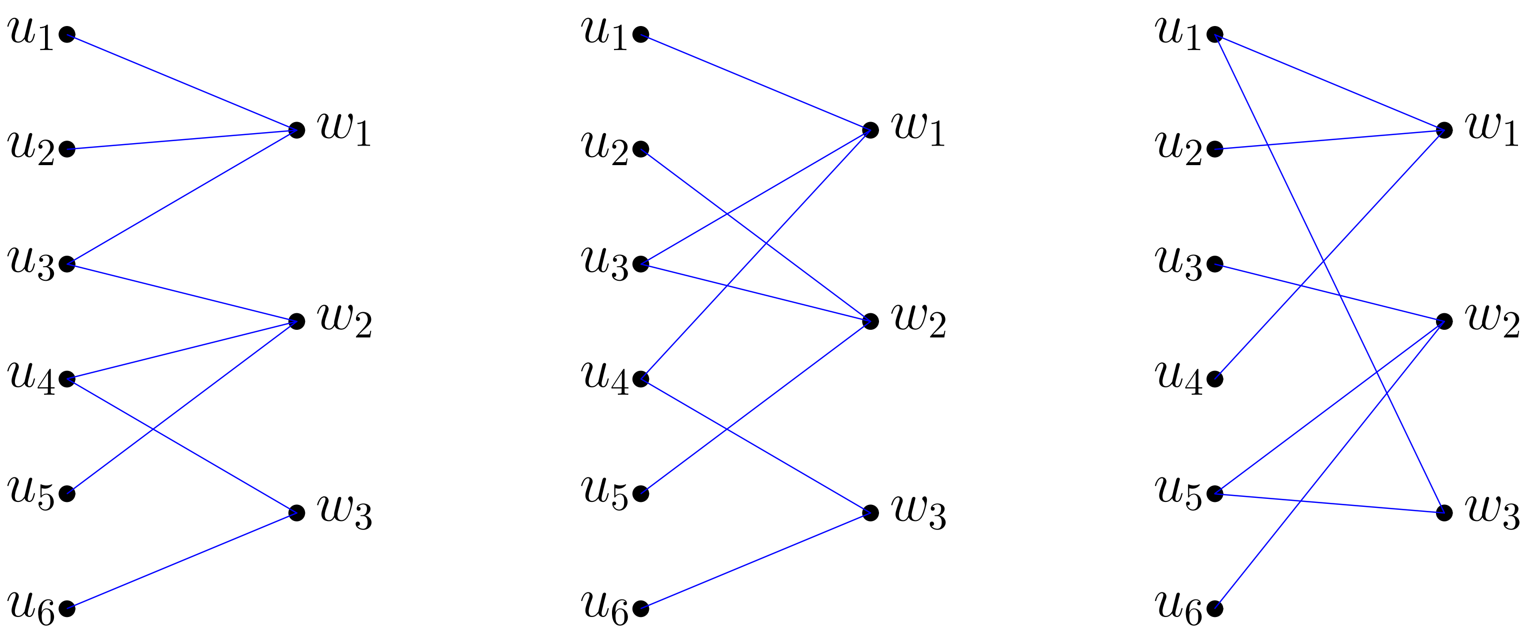
The summation of the above formula is generally complicated. However, in some special cases, we get nice simple descriptions. The family of *central transportation polytopes of order $kn \times n$* is such an example.

$ST_{k,n}$

Definition 18. Let $ST_{k,n}$ be the set of spanning trees of the complete bipartite graph $K_{kn,n}$ in which the vertices $w_1, w_2, \dots, w_n$ have degree $k+1, k+1, \dots, k+1, k$.

Example: Let $k = 2$ and $n = 3$. Below are three trees in $ST_{2,3}$.



Lemma 19. *The cardinality of* $ST_{k,n}$ *is* $\dfrac{(kn)!}{(k!)^n} n^{n-2} k^{n-1}$.

The MGF of a central transportation polytope of order $kn \times n$

Theorem 20 (L.). *Assume that $\mathcal{T}(\mathbf{r}, \mathbf{c})$ is an integral central transportation polytope of order $kn \times n$. (Hence, $\mathbf{r} = (a, \dots, a)$ and $\mathbf{c} = (b, \dots, b)$ are two integer vectors, where $akn = bn$.)*

Then the set $ST_{k,n}$ is precisely the set of spanning trees of $K_{kn,n}$ that admits a positive labelling with margin $(\mathbf{r}(t), \mathbf{c}(t))$.

Therefore,

$$f(\mathcal{T}(\mathbf{r}, \mathbf{c}), \mathbf{z}) = \sum_{T \in ST_{k,n}} \mathbf{z}^{M_T} \prod_{e \notin E(T)} \frac{1}{1 - \mathbf{z}^{\text{cycle}(T,e)}}. \tag{21}$$

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Remark 22. When $k = 1$, the central transportation polytopes of order $n \times n$ are the Birkhoff polytopes. Our theorem recover the combinatorial formula for the MGF of the Birkhoff polytopes obtained by my joint work with De Loera and Yoshida.

Maximum number of vertices

Definition 23. We define $\phi(m, n)$ to be the maximum possible number of vertices of all transportation polytopes of order $m \times n$.

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Corollary 25. *Suppose $\mathcal{T}(\mathbf{r},\mathbf{c})$ is a central transportation polytope of order $m \times n$. Hence, $\mathbf{r} = (a,a,\dots,a)$ and $\mathbf{c} = (b,b,\dots,b)$ where $am = bn$. In this case, we have*

$$\mathbf{r}(t) = (a-t,\dots,a-t), \ \mathbf{c}(t) = (b,\dots,b,b-mt).$$

Then for sufficiently small t, the perturbed transportation polytope $\mathcal{T}(\mathbf{r}(t),\mathbf{c}(t))$ has $\phi(m,n)$ vertices.

Maximum number of vertices (cont'd)

Corollary 26. *Assume the same conditions as the last corollary. Then* $\phi(m,n)$ *is the number of spanning trees of* $K_{m,n}$ *that admits a positive labelling with margin* $(\mathbf{r}(t), \mathbf{c}(t))$.

Maximum number of vertices (cont'd)

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$$\phi(kn,n) = |ST_{k,n}| = \frac{(kn)!}{(k!)^n} n^{n-2} k^{n-1}$$

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Remark 28. Suppose $\mathcal{T}(\mathbf{r},\mathbf{c})$ is a central transportation polytope of order $m \times n$. studying spanning trees of $K_{m,n}$ that admits a positive labelling with margin $(\mathbf{r}(t),\mathbf{c}(t))$ can lead to:

- A combinatorial formula for the MGF of $\mathcal{T}(\mathbf{r},\mathbf{c})$.
- An explicit formula for $\phi(m,n)$.