# Uniqueness of Berline-Vergne's valuation

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This is joint work with Federico Castillo.

# Outline

- Introduction
  - Polytopes and Ehrhart theory
  - McMullen's formula and $\alpha$-constructions
- Main example and the uniqueness result
- Idea of proof

## Polytopes and Ehrhart polynomials

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**Theorem** (Ehrhart)**.** *Let $P$ be a $d$-dimensional integral polytope. Then $i(P, t)$ is a polynomial in $t$ of degree $d$.*

Therefore, we call $i(P, t)$ the *Ehrhart polynomial* of $P$.

# McMullen's formula

In 1975 Danilov asked if it is possible to assign values $\Psi(C)$ to all rational cones $C$ such that the following *McMullen's formula* holds

$$\mathrm{Lat}(P) = \sum_{F\text{: a face of } P} \alpha(F, P)\,\mathrm{vol}(F).$$

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McMullen was the first to prove that it was possible in a non-constructive way.

## A refinement of Ehrhart coefficients

Applying McMullen's formula to the dilation $tP$ of $P$, we obtain

$$\begin{aligned} i(P,t) = \mathrm{Lat}(tP) &= \sum_{F\text{: a face of } P} \alpha(tF, tP)\,\mathrm{vol}(tF) \\ &= \sum_{F\text{: a face of } P} \alpha(F, P)\,\mathrm{vol}(F) t^{\dim(F)} \end{aligned}$$

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Conclusion. Constructions for $\Psi/\alpha$ are helpful in understanding the coefficients of Ehrhart polynomials.

# Different Constructions

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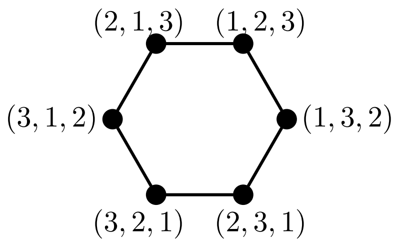
Questions:

Is it possible that under certain constraint, the $\Psi/\alpha$ construction for McMullen's formula is unique?

Can we find better ways to compute $\alpha$-values?

## Main example

Example. Suppose $\Psi$ is a solution to the McMullen's formula.

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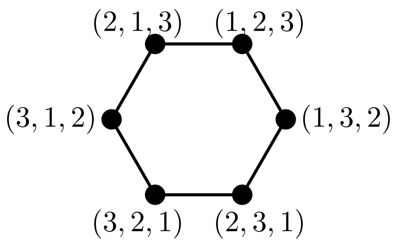


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The pointed feasible cones of the six vertices are

$$\mathrm{Cone}((-1,1,0),(1,0,-1)), \quad \mathrm{Cone}((-1,0,1),(0,1,-1)), \quad \mathrm{Cone}((-1,1,0),(-1,1,0)),$$

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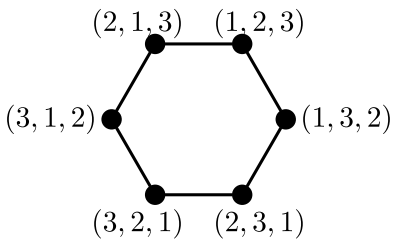


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$$1 = \sum_{v:\text{a vertex of } P} \alpha(v,P)\,\mathrm{vol}(v) = \alpha_1 + \cdots + \alpha_6.$$

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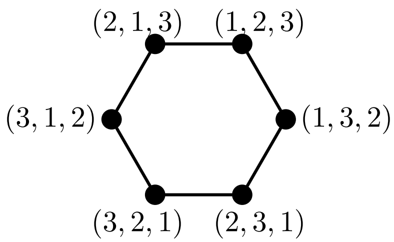


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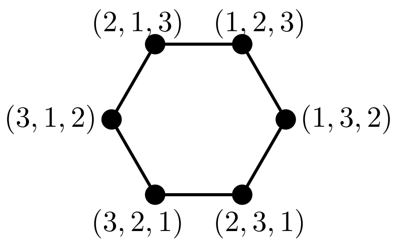


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We will use Berline-Vergne's construction, which we will refer to as the *BV-construction* or *BV-$\alpha$-valuation*.

## Berline-Vergne's construction

**Important facts** about the BV-construction:

- Certain valuation property.
- Invariant under $O_n(\mathbb{Z})$ – orthogonal unimodular transformations, in particular *invariant under rearranging coordinates with signs*.

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**Definition.** We say a solution $\Psi$ to the McMullen's formula is *symmetric about the coordinates*, if for any cone $C \in \mathbb{R}^n$ and any signed permutation $(\sigma, \mathbf{s}) \in \mathfrak{S}_n \times \{\pm 1\}^n$, we have

$$\Psi(C) = \Psi((\sigma, \mathbf{s})(C)),$$

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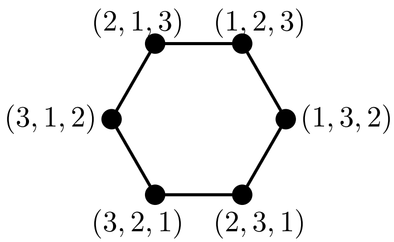


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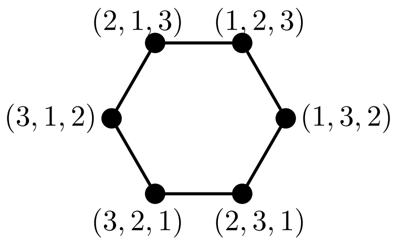


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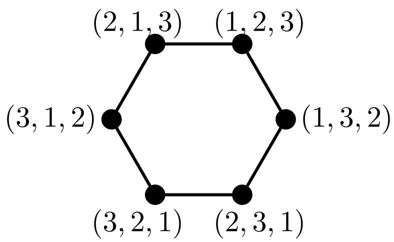


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Solving this, we get $\alpha = 1/6$.

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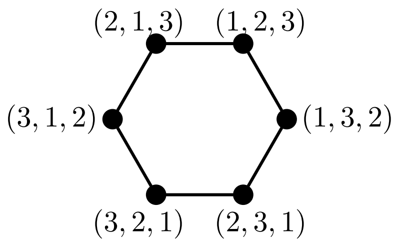


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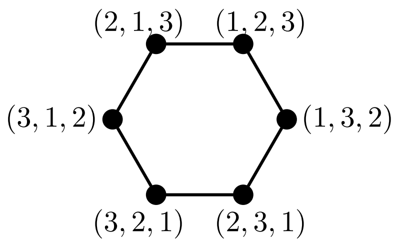


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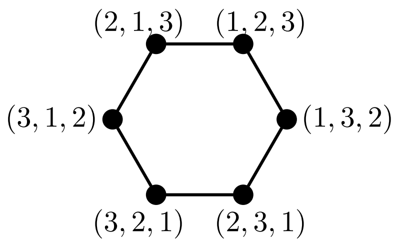


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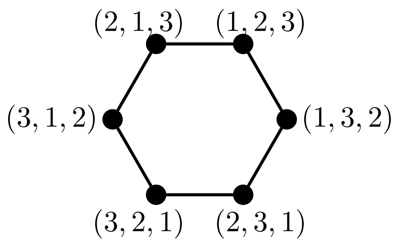


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Solving it, we get $\beta = 1/2$.

## Observation

When we figured out the $\Psi/\alpha$ values in our example in the previous slides:

- We did not apply Berline-Vergne's original recursive construction.
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Hence, $\Psi$ of the cones or equivalently the values of $\alpha(F, P)$ appeared in our example are **unique**.

## Uniqueness question of the construction of $\Psi$

Is it true that $\Psi$ in McMullen's formula is uniquely determined if we require it to be symmetric about the coordinates?

If so, then the BV-construction is the **only** symmetric construction.

## Permutohedra and uniqueness result

**Definition.** Suppose $\mathbf{v} = (v_1, v_2, \cdots, v_d, v_{d+1})$ is a strictly increasing sequence. We define the *usual permutohedron*

$$\mathrm{Perm}\,(\mathbf{v}) := \mathrm{conv}\left\{\left(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(d)}, v_{\sigma(d+1)}\right) \ : \ \sigma \in \mathfrak{S}_{d+1}\right\}.$$

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**Theorem** (Castillo-L.)**.** *Suppose $\Psi$ is a solution to McMullen's formula and is symmetric about the coordinates. Then the values of $\Psi$ on cones arising from usual/regular permutohedra are uniquely determined.*

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**Theorem** (Castillo-L.)**.** *Suppose* $\Psi$ *is a solution to McMullen's formula and is symmetric about the coordinates. Then the values of* $\Psi$ *on cones arising from usual/regular permutohedra are uniquely determined.*

*Idea of proof:* Describe $\mathrm{Lat}(\mathrm{Perm}(\mathbf{v}))$ in two ways: Use mixed Ehrhart theory and use McMullen's formula.

# Mixed Ehrhart Theorem

Consider the following Minkwoski sum:

$$P = w_1 P_1 + w_2 P_2 + \cdots + w_k P_k,$$

where $w_i$ are variables and $P_i$ are polytopes.

**Mixed Ehrhart Theorem** Suppose $P_1, \cdots, P_k$ are integral polytopes satisfying $\dim(P_1 + \cdots + P_k) = d$. Then the number of lattice points in $P$ is a polynomial in $w_i$'s of degree $d$. More precisely,

$$\mathrm{Lat}(w_1 P_1 + w_2 P_2 + \cdots + w_k P_k) = \sum_{e=0}^{d} \sum_{j_1, \cdots, j_e = 1}^{k} \mathcal{M}\mathrm{Lat}^e(P_{j_1}, P_{j_2}, \cdots, P_{j_e}) w_{j_1} \cdots w_{j_e},$$

where the coefficients $\mathcal{M}\mathrm{Lat}^e(P_{j_1}, P_{j_2}, \cdots, P_{j_e})$ are fixed numbers, called *mixed Ehrhart coefficients*.

## Mixed Ehrhart Theorem (cont'd)

Postnikov showed that usual permutohedra are Minkowski sums of hypersimplices.

$$\mathrm{Perm}(\mathbf{v}) = w_1 \Delta_{1,d+1} + w_2 \Delta_{2,d+1} + \cdots + w_d \Delta_{d,d+1},$$

where

$$w_i := v_{i+1} - v_i \text{ for } i = 1, 2, \ldots, d,$$

and the *hypersimplex* $\Delta_{k,d+1}$ is defined as

$$\Delta_{k,d+1} = \mathrm{Perm}(\underbrace{0, \cdots, 0}_{d+1-k}, \underbrace{1, \cdots, 1}_{k}).$$

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and the *hypersimplex* $\Delta_{k,d+1}$ is defined as

$$\Delta_{k,d+1} = \mathrm{Perm}(\underbrace{0, \cdots, 0}_{d+1-k}, \underbrace{1, \cdots, 1}_{k}).$$

Hence, applying Mixed Ehrhart theory,

$$\mathrm{Lat}(\mathrm{Perm}(\mathbf{v})) = \sum_{e=0}^{d} \sum_{j_1, \cdots, j_e = 1}^{d} \mathcal{M}\mathrm{Lat}^e(\Delta_{j_1,d+1}, \Delta_{j_2,d+1}, \cdots, \Delta_{j_e,d+1}) w_{j_1} \cdots w_{j_e}$$

is a polynomial in $w_i$'s, which we denote by $E(w_1, \ldots, w_d)$.

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$$\mathrm{Lat}(\mathrm{Perm}(\mathbf{v})) = \sum_{F} \alpha(F, P)\,\mathrm{vol}(F) = \sum_{S \subseteq [d]} O_d(S)\alpha_d(S)\,\mathrm{vol}(F_S),$$

where $O_d(S)$ and $\alpha_d(S)$ are *independent* from the choices of $\mathbf{v}$.

Only $\mathrm{vol}(F_S)$ *varies* when $\mathbf{v}$ changes.

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Hence, we obtain another description for $E(w_1, \dots w_d)$.

## A formula

Comparing the coefficient of $w_S := \prod_{i \in S} w_i$ in the two expressions for $E(w_1, \dots, w_d)$, we obtain the following:

**Theorem** (Castillo-L.)**.** *Suppose $\Psi$ is a solution to McMullen's formula and is symmetric about the coordinates. Then the $\alpha$ values for the regular permutohedron $\Pi_d$ are positive scalars of mixed Ehrhart coefficients of hypersimplices.*