Reflexive Simplices with Ehrhart h^*-Polynomial Roots on the Unit Circle

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This is joint work with Ben Braun.

h^*-unit-circle-rootedness

We say a lattice polytope P is *h^*-unit-circle-rooted* if all the roots of its (Ehrhart) h^*-polynomial are on the unit circle $\{z : |z| = 1\}$ in the complex plane.

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Why do I care?

$$h^*\text{-unit-circle-rootedness} \quad \Longrightarrow \quad \text{Ehrhart positivity}$$

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h^*-unit-circle-rootedness $\implies$ Ehrhart positivity

Another fact

h^*-unit-circle-rootedness and P has an interior lattice point

$\implies$ P is reflexive

Question

Which reflexive polytopes are h^*-unit-circle-rooted?

The reflexive simplex $\Delta_{(1,\boldsymbol{q})}$

Given a vector of positive integers $\boldsymbol{q} \in \mathbb{Z}^d$, we define

$$\Delta_{(1,\boldsymbol{q})} := \operatorname{conv}\left\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d, -\sum_{i=1}^{d} q_i \boldsymbol{e}_i\right\}.$$

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We describe $\boldsymbol{q}$ by its *support* and *multiplicity*, e.g., $\boldsymbol{q} = (2,2,2,2,3,3,3) = (2^4, 3^3)$ has support $\boldsymbol{r} = (2,3)$ with multiplicity $\boldsymbol{x} = (4,3)$.

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- Conrads gives a characterization for when $\Delta_{(1,\boldsymbol{q})}$ is reflexive.
- Using Conrads' condition, one can show that if $\boldsymbol{q} = (r^x)$ is supported on one integer r, then $\Delta_{(1,\boldsymbol{q})}$ is reflexive if and only if $r = 1$. This gives us the *standard reflexive simplex*, which is h^*-unit-circle-rooted.

q-vectors supported on two integers

Theorem 1 (Braun-L.). *Suppose* $\boldsymbol{q}$ *is supported on two integers* $\boldsymbol{r} = (r_1, r_2)$ *with multiplicity* $\boldsymbol{x} = (x_1, x_2)$ *satisfying Conrads' condition for reflexivity. Then* $\Delta_{(1,\boldsymbol{q})}$ *is* h^**-unit-circle-rooted if* $\boldsymbol{r}$ *and* $\boldsymbol{x}$ *are in one of the following three forms:*

i. $\boldsymbol{r} = (a, ka-1)$ *and* $\boldsymbol{x} = ((ka-1)c-k, a((ka-1)c-k)+1)$.

ii. $\boldsymbol{r} = (a, a-1)$ *and* $\boldsymbol{x} = ((a-1)c-1, ac+1)$.

iii. $\boldsymbol{r} = (a, a^2-1)$ *and* $\boldsymbol{x} = ((a^2-1)c-a, a(ac-1)+1)$.

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Conjecture 2 (Braun-L.). *For any fixed $\boldsymbol{r} = (r_1, r_2)$, among all choices of $\boldsymbol{x} = (x_1, x_2)$ such that $\Delta_{(1,\boldsymbol{q})}$ is reflexive and h^*-unit-circle-rooted, all but finitely many fit into the three situations described by the theorem above.*