Isometric Immersions into Hyperbolic 3-Space

Andrew Gallatin

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Mathematics Department
California Polytechnic State University
San Luis Obispo
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AUTHOR: Andrew Gallatin

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Senior Project Advisor        Signature

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Mathematics Department Chair  Signature
## Contents

1 **Geometry of Manifolds** ................................................................. 2  
1.1 Review of Riemannian Manifolds ................................................. 2  
1.2 Tensors and the First Fundamental Form ...................................... 4  
1.3 Levi-Civita Connection and Christoffel Symbols .......................... 7  
1.4 Riemann Curvature Tensor ......................................................... 12  
1.5 Immersions, Embeddings, and Isometries ..................................... 14  
1.6 Second Fundamental Form ......................................................... 16  
1.7 Gauss and Codazzi-Mainardi Equations .................................... 19  
1.8 Fundamental Theorem of Surfaces ............................................ 22  

2 **Hyperbolic 3-Space** ................................................................. 25  
2.1 $\mathbb{H}^3$ as a Subset of Minkowski Spacetime .......................... 25  
2.2 Poincaré Ball Model ................................................................. 26  
2.3 Upper Half Space Model ............................................................ 28  
2.4 Geometry of $\mathbb{H}^3$ ............................................................. 29  
2.5 Geometry of Horospheres ......................................................... 30  
2.6 Geometry of Equidistant Cylinders .......................................... 31  

3 **Isometric Immersion of $\mathbb{R}^2$ into $\mathbb{H}^3$**  ...................... 34  
3.1 Volkov and Vladimirova’s Result .............................................. 34
Chapter 1

Geometry of Manifolds

1.1 Review of Riemannian Manifolds

Definition (Coordinate Patch). Let $D \subseteq \mathbb{R}^n$ be open and let $M$ be a set. A coordinate patch in $M$ is a one-to-one function $x : D \to M$.

Definition (Differentiable Manifold). An $n$-dimensional differentiable manifold is a set $M$ and a collection $\mathcal{P}$ of coordinate patches $x : D \subseteq \mathbb{R}^n \to M$ that satisfy the following properties:

(a) (The covering axiom) For each point $p \in M$, there is a coordinate patch $x : D \subseteq \mathbb{R}^n \to M$ such that $p \in x(D)$.

(b) (The smooth overlap axiom) For any coordinate patches $x, y \in \mathcal{P}$ whose images intersect non-trivially, the composite functions $x^{-1} \circ y$ and $y^{-1} \circ x$ are Euclidean differentiable where defined.

(c) (The Hausdorff axiom) For all points $p, q \in M$ with $p \neq q$, there exist patches $x : D \to M$ and $y : E \to M$ with $p \in x(D)$, $q \in y(E)$, and $x(D) \cap y(E) = \emptyset$.

Figure 1.1: A visualization of the smooth overlap axiom for manifolds.
It is worth noting that a manifold is typically defined using the charts instead of coordinate patches. In
the above definition, a \textit{chart} is the inverse of a coordinate patch whose range is restricted to its image. In
other words, for each \( x \in \mathcal{P} \), the corresponding chart is \( x^{-1} : \mathcal{D} \to \mathcal{D} \). The collection of charts on \( M \) is
referred to as an \textit{atlas}.

For a more intuitive visualization of the smooth overlap axiom, see Figure 1.1. We can endow our
manifold with additional structure by requiring various properties on \( x^{-1} \circ y \) and \( y^{-1} \circ x \). We can therefore
classify different manifold structures by the properties of \( x^{-1} \circ y \) and \( y^{-1} \circ x \) as seen in Table 1.1.

\[
\begin{array}{ccc}
\text{\( x^{-1} \circ y \) and \( y^{-1} \circ x \) are:} & \text{M is said to be:} \\
\text{continuous} & \Rightarrow & \text{topological manifold} \\
\text{differentiable} & \Rightarrow & \text{differentiable manifold} \\
\text{\( C^1 \)-differentiable} & \Rightarrow & \text{\( C^1 \)-manifold} \\
\text{\( C^k \)-differentiable} & \Rightarrow & \text{\( C^k \)-manifold} \\
\text{\( C^\infty \)-differentiable} & \Rightarrow & \text{\( C^\infty \)-manifold} \\
\text{real analytic} & \Rightarrow & \text{real analytic manifold} \\
\text{orientation-preserving} & \Rightarrow & \text{orientable manifold} \\
\end{array}
\]

Table 1.1: Additional structures on manifolds.

For the rest of this discussion, we take manifold to mean a \( C^\infty \)-manifold unless otherwise stated.

Recall that we have a well-defined notion of the tangent space to a point \( p \in M \) defined by the initial
velocity vectors of curves passing through \( p \). We denote the tangent space at \( p \) as \( T_p M \). Given a coordinate
patch \( x \) with coordinates \( x^1, \ldots, x^n \in \mathbb{R}^n \), we can consider the image of parameter curves in \( \mathbb{R}^n \) under \( x \). For
a parameter curve along \( x^i \), we denote the tangent vector at \( p \in M \) that lies tangent to the image of the
parameter curve as \( \frac{\partial}{\partial x^i} \). If we have a specific coordinate patch \( x : D \subseteq \mathbb{R}^n \to M \), we can calculate
\[
\frac{\partial}{\partial x^i} = \frac{\partial x}{\partial x^i} = x^i.
\]

Note that this notation makes sense when compared to the partial derivative notation. If \( f \) is a scalar
function, recall that a vector \( v \) acts on \( f \) by taking the directional derivative of \( f \) in the direction of \( v \). This
is denoted by \( v[f] = D_v(f) \). So for \( f : M \to \mathbb{R} \), \( \frac{\partial}{\partial x^i}[f] \) is precisely the directional derivative of \( f \) in the
direction \( \frac{\partial}{\partial x^i} \). In the Euclidean case where \( M = \mathbb{R}^n \), this is equivalent to the partial derivative of \( f \) with
respect to \( x^i \), where it is commonly understood that \( \frac{\partial}{\partial x^i}[f] = \frac{\partial}{\partial x^i}f \).

This leads to the following theorem which we will state but not prove.

\textbf{Theorem 1.} Let \( M \) be an \( n \)-dimensional differentiable manifold. Then for each point \( p \in M \) with a
coordinate patch \( x \) and coordinates \( x^1, \ldots, x^n \), the tangent space \( T_p M \) is an \( n \)-dimensional real vector
space. Moreover,
\[
\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}
\]
is a basis for \( T_p M \). Moreover, for any \( v \in T_p M \), we have
\[
v = \sum_{i=1}^{n} v [x^i] \frac{\partial}{\partial x^i}.
\]

Furthermore, we can define a vector field \( X \) on \( M \) to be a map that assigns to each point \( p \in M \) a
tangent vector \( X(p) \in T_p M \). We also require that the vector field is smooth (\( C^\infty \)) when varying \( p \in M \). For
the rest of this exposition, it is understood that all vector fields are smooth.

We are now able to endow the tangent spaces to a manifold with inner product structure in such a way
to allow for calculus on the manifold.
**Definition** (Riemannian Metric). A collection of maps $g_p : T_p M \times T_p M \to \mathbb{R}$ on a manifold $M$ is a Riemannian metric tensor if

1. given a point $p \in M$, $g_p$ is an inner product.
2. for all differentiable vector fields $V, W$ on $M$, $g_p(V(p), W(p))$ is a differentiable real-valued function.

We will commonly refer to the Riemannian metric tensor as $g$. Given a coordinate patch with coordinates $x^1, ..., x^n$, we write the matrix components of $g$ as $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ for $i, j = 1, ..., n$. Thus we can represent $g$ as an $n \times n$ matrix $(g_{ij})$. Note that without specifying a basepoint, the matrix components are differentiable functions waiting to evaluate on $p \in M$.

**Definition** (Riemannian Manifold). A manifold $M$ endowed with a Riemannian metric tensor $g$ is called a Riemannian manifold. We often denote a Riemannian manifold by the pair $(M, g)$.

Given a differentiable map between manifolds, there is a natural way to define corresponding maps between the tangent spaces of each manifold.

**Definition** (Differential). Let $M$ and $N$ be differentiable manifolds and $\phi : M \to N$ be a differentiable map. For $p \in M$ and $q \in N$ with $\phi(p) = q$, the differential of $\phi$ at $p$ is defined as the map $d\phi : T_p M \to T_q N$

such that if $v$ is the initial velocity vector of a curve $\alpha$ in $M$, then $d\phi(v)$ is the initial velocity of the image curve $F(\alpha)$ in $N$.

### 1.2 Tensors and the First Fundamental Form

Given a manifold $M$ and a local coordinate patch $x$ with coordinates $x^1, ..., x^n$, it is common to denote the Euclidean derivatives as

$$\frac{\partial x}{\partial x^i} = x_i.$$

Furthermore, we can denote the second derivatives by appending a subscript index:

$$\frac{\partial^2 x}{\partial x^i \partial x^j} = x_{ij}.$$

Hence, for vector-valued functions, we adopt superscripts to dictate the component functions and subscripts to dictate derivatives taken with respect to the $i$-th coordinate of a vector-valued function.

For the remainder of this exposition, we will use Einstein notation to simplify sums. This notation is used widely in Ricci calculus and is prevalent throughout differential geometry. Einstein notation is used to condense formulas that are expressed as a sum over an index that appears in the subscript of one variable and the superscript of another. The summation symbol is then suppressed and the formula is assumed to be summed over the index that appears as both a subscript and superscript.
**Example 1.** Consider a vector-valued function $f : \mathbb{R} \rightarrow \mathbb{R}^n$. If we consider the coordinates $x^1, ..., x^n$ to be the coordinates in $\mathbb{R}^n$ and $f^1, ..., f^n$ to be the component functions of $f$, we can express the divergence of $f$ as

$$\nabla \cdot f = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} f^i.$$ 

In Einstein notation, we suppress the summation symbol and it is assumed that we sum over the index $i$:

$$\nabla \cdot f = \frac{\partial}{\partial x^i} f^i.$$ 

**Example 2.** Let $g$ be a Riemannian metric tensor. Given local coordinates $x^1, ..., x^n$, we can express $g$ as a matrix $(g_{ij})$ where $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. Treating $(g_{ij})$ is an $n \times n$ matrix, we denote the components of the inverse matrix by $g^{ij}$. As the inverse, we have $(g^{ij})(g_{ij}) = I_{n \times n}$. Using the Kronecker-delta function $\delta^i_j$, we can rewrite $I = (I_{ij}) = (\delta^i_j)$. Then matrix multiplication gives

$$\sum_{k=1}^{n} g_{ik} g^{kj} = \delta^i_j.$$ 

In Einstein notation, this is written cleanly as

$$g_{ik} g^{kj} = \delta^i_j.$$ 

Similarly since $(g^{ij})(g_{ij}) = I_{n \times n}$ we have

$$g^{ik} g_{kj} = \delta^i_j.$$ 

We can now generalize the notion of a metric tensor to arrive at a characterization of tensors.

**Definition.** A **covariant** $(0,n)$-**tensor** at point $p \in M$ is a multi-linear map

$$\Phi : T_p M \times \cdots \times T_p M \rightarrow \mathbb{R}.$$ 

Given coordinates $x^1, ..., x^n$, we denote the components of $\Phi$ as

$$\Phi_{i_1, ..., i_n} = \Phi\left(\frac{\partial}{\partial x^{i_1}}, ..., \frac{\partial}{\partial x^{i_n}}\right).$$ 

A $(1,n)$-**tensor** at point $p \in M$ is a multi-linear map

$$\Phi : T_p M \times \cdots \times T_p M \rightarrow T_p M.$$ 

Given coordinates $x^1, ..., x^n$,

$$\sum_j \Phi_{i_1, ..., i_n}^{j} \frac{\partial}{\partial x^j} = \Phi\left(\frac{\partial}{\partial x^{i_1}}, ..., \frac{\partial}{\partial x^{i_n}}\right).$$ 

We say that a tensor is **differentiable** if the component functions $\Phi_{i_1, ..., i_n}$ or $\Phi_{i_1, ..., i_n}^j$ are differentiable as the basepoint $p \in M$ is varied.
**Example 3.** Clearly any Riemannian metric is a \((0,2)\)-tensor. A vector field \(X\) on a manifold \(M\) is a \((1,0)\)-tensor with \(X = \sum_i X^i \frac{\partial}{\partial x^i}\) given coordinates \(x^1,\ldots,x^n\). Any one-form defined on \(M\) is a \((0,1)\)-tensor, while a two-form is a \((0,2)\)-tensor.

Suppose \((M,g)\) is a Riemannian manifold with local coordinates \(x^1,\ldots,x^n\). Given a \((1,n)\)-tensor \(\Phi\) on \(M\) with components \(\Phi_{i_1\ldots i_n}^{j}\), we can “lower indices” by applying the metric \(g\):

\[
\Phi_{i_1\ldots i_n}^{j} = g_{kj} \Phi_{i_1\ldots i_n}^{j}.
\]

In other words, given a \((1,n)\)-tensor \(\Phi\),

\[
g \left( \Phi_{i_1\ldots i_n}^{j} \cdot, \cdot \right) : T_p M \times \cdots \times T_p M \to \mathbb{R}
\]

is a \((0,n+1)\)-tensor. Conversely, to “raise indices”, we apply the inverse \(g^{-1}\):

\[
\Phi_{i_1\ldots i_n}^{j} = \Phi_{i_1\ldots i_n}^{j} g^{kj}.
\]

This process of raising and lowering indices will allow us to find principal curvatures from the second fundamental form by transforming the second fundamental form into the shape operator. Note that we take the convention of raising and lowering the last index. We can now use our definition of tensors to define the first fundamental form for a Riemannian manifold.

**Definition (First Fundamental Form).** Given a Riemannian manifold \((M,g)\), the **first fundamental form** is the \((0,2)\)-tensor given by

\[
I(x,y) = g(x,y).
\]

Given coordinates \(x^1,\ldots,x^n\), the coefficients of \(I\) are given by

\[
I_{ij} = g_{ij} = g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right).
\]

**Example 4.** Consider a sphere \(\Sigma \subseteq \mathbb{R}^3\) with radius \(R > 0\). Here, \(g\) is the usual Euclidean metric (dot product) on \(\mathbb{R}^3\) restricted to \(\Sigma\). We can parametrize \(\Sigma\) using one coordinate patch:

\[
x(\theta, \phi) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi)
\]

for \(\theta \in [0,2\pi]\) and \(\phi \in [0,\pi]\). Then

\[
\frac{\partial}{\partial \theta} = (-R \sin \theta \sin \phi, R \cos \theta \sin \phi, 0)
\]

\[
\frac{\partial}{\partial \phi} = (R \cos \theta \cos \phi, R \sin \theta \cos \phi, -R \sin \phi)
\]

Hence,

\[
I = \begin{pmatrix}
I_{11} & I_{12} \\
I_{21} & I_{22}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial \phi} \cdot \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \cdot \frac{\partial}{\partial \phi}
\end{pmatrix} = \begin{pmatrix}
R^2 \sin^2 \phi & 0 \\
0 & R^2
\end{pmatrix}.
\]
Example 5. Consider the hyperbolic plane $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with metric $g = \frac{1}{y^2} (dx^2 + dy^2)$. Then at any point $(x, y) \in \mathbb{H}^2$,

$$I = g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$ 

Note that for any metric written $g = E(p)dx^2 + F(p)dxdy + G(p)dy^2$, we have

$$I = g = \begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix}.$$

$\diamondsuit$

1.3 Levi-Civita Connection and Christoffel Symbols

For surfaces in $\mathbb{R}^3$, the covariant derivative is used to differentiate both scalar functions and vector fields. For vector fields $X$ and $Y$, $\nabla_X Y$ is defined to be the tangential component of the directional derivative of $Y$ in the direction $X$. The Levi-Civita connection, or Riemannian connection, is the generalization of the covariant derivative for Riemannian manifolds. In other words, the Levi-Civita connection is our tool for differentiating both scalar functions and vector fields on a given a manifold. More importantly, the Levi-Civita connection will allow us to define the Christoffel symbols which we will use extensively in computations.

**Definition (Lie Bracket).** Let $M$ be a Riemannian manifold and let $X$ and $Y$ be vector fields on $M$. For differentiable functions $f : M \to \mathbb{R}$, define

$$[X, Y](f) = X[Y[f]] - Y[X[f]].$$

Then the Lie bracket of $X$ and $Y$ is the vector field $[X, Y]$ on $M$. Given a point $p \in M$,

$$[X, Y](p)(f) = X(p)[Y[f]] - Y(p)[X[f]].$$

If $f$ is a scalar function, we can extend the definition of the Lie bracket by defining

$$[X, f] = X[f].$$

Note that the Lie bracket is defined entirely through directional derivatives and does not require a metric on $M$. In a sense, the Lie bracket measures how two vector fields commute via directional derivatives. From this definition, we have the following properties.

**Proposition 2 (Properties of the Lie Bracket).** Let $M$ be a Riemannian manifold with vector fields $X, Y, Z$. Let $\alpha, \beta \in \mathbb{R}$ and $f, h : M \to \mathbb{R}$ be differentiable. Then

1. $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z].$
2. $[X, Y] = -[Y, X].$
4. Given coordinates $x^1, \ldots, x^n$, $\left[ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^n} \right] = 0.$
Proof. Note that most of these properties will follow directly from the properties of the directional derivative.

1. For all differentiable \( \phi : M \to \mathbb{R} \),

\[
\left[ \alpha X + \beta Y, Z \right](\phi) = (\alpha X + \beta Y)[Z(\phi)] - Z[(\alpha X + \beta Y)(\phi)]
\]

(Linearity of directional derivative) \( = \alpha X[Z(\phi)] + \beta Y[Z(\phi)] - \alpha Z[X(\phi)] - \beta Z[Y(\phi)] \)

\( = \alpha[X, Z](\phi) + \beta[Y, Z](\phi). \)

2. For all differentiable \( \phi : M \to \mathbb{R} \),

\[
[X, Y](\phi) = X[Y(\phi)] - Y[X(\phi)] = -[Y, X](\phi).
\]

3. For all differentiable \( \phi : M \to \mathbb{R} \),

\[
[f X, h Y](\phi) = f X[h Y(\phi)] - h Y[f X(\phi)]
\]

(Product rule of directional derivative) \( = f X[h Y(\phi)] + f h X[Y(\phi)] - h Y[f X(\phi)] - f h Y[X(\phi)] \)

\( = f h[X, Y](\phi) + f X[h Y(\phi)] - h Y[f X(\phi)]. \)

4. Suppose \( x^1, ..., x^n \) are coordinates for a patch on \( M \). Then for all differentiable \( \phi : M \to \mathbb{R} \), note that

\[
\frac{\partial}{\partial x^i} \left( \frac{\partial \phi}{\partial x^j} \right) = \frac{\partial^2 \phi}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^j} \left( \frac{\partial \phi}{\partial x^i} \right).
\]

Thus, since the directional derivatives of coordinate tangent vectors commute, \( \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0. \)

Now we can generalize the covariant derivative from surfaces in \( \mathbb{R}^3 \) to make sense for an arbitrary Riemannian manifold. The idea is to define an operator that associates to two vector fields another vector field with all of the properties that the covariant derivative enjoyed for surfaces. Since we desire this operator to “behave nicely” with the Riemannian metric, we cannot simply use the directional derivative to define the connection like we did for the covariant derivative. By similar reasoning, we cannot take the Lie bracket to be our generalized covariant derivative because it does not have a sense of the metric. Thus, we resort to defining the Levi-Civita connection through desired properties. As we will see, given a Riemannian manifold, the metric uniquely determines this connection.

**Definition (Levi-Civita Connection).** Let \( (M, g) \) be a Riemannian manifold. Let \( X, Y \) be vector fields on \( M \). The **Levi-Civita connection** is a map taking \( X \) and \( Y \) to a vector field \( \nabla_X Y \) satisfying the following properties:

1. \( \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z. \)
2. \( \nabla_{fX} Y = f \nabla_X Y. \)
3. \( \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z. \)
4. \( \nabla_X (fY) = f \nabla_X Y + X[f]Y. \)
5. \( X[g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \)
6. \( \nabla_X Y - \nabla_Y X - [X, Y] = 0. \)

Notice that the covariant derivative for surfaces in \( \mathbb{R}^3 \) satisfies the connection properties and is the Levi-Civita connection for the Euclidean metric. Also observe that property 6 gives another characterization of the Lie bracket:

\[
[X, Y] = \nabla_X Y - \nabla_Y X.
\]

We will now show that for any given \( (M, g) \), there is a uniquely determined Levi-Civita connection \( \nabla \) on \( M \).
Theorem 3 (Uniqueness and Existence of the Levi-Civita Connection). Let \( (M, g) \) be a Riemannian manifold. Then there exists a unique Levi-Civita connection \( \nabla \) on \( M \).

**Proof.** Suppose \( (M, g) \) is a Riemannian manifold. We will first derive the Koszul formula for the Levi-Civita connection. If \( \nabla_X Y \) is a Levi-Civita connection on \( M \), by property 5, we have for an arbitrary vector field \( Z \) on \( M \),

\[
\begin{align*}
X[g(Y, Z)] &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad (1.1) \\
Y[g(X, Z)] &= g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \quad (1.2) \\
Z[g(X, Y)] &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \quad (1.3)
\end{align*}
\]

Calculating \( (1.1) + (1.2) - (1.3) \) and recalling that \( [X, Y] = \nabla_X Y - \nabla_Y X \), we have

\[
X[g(Y, Z)] + Y[g(X, Z)] - Z[g(X, Y)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(Y, \nabla_Y Z) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = g(X, \nabla_Y Z - \nabla_Z Y) + g(Y, \nabla_X Z - \nabla_X Z) + g(Z, \nabla_X Y + \nabla_Y X) = g(X, [Y, Z]) + g(Y, [X, Z]) + g(Z, 2\nabla_X Y + [Y, X]).
\]

Rearranging, we recover the Koszul formula

\[
2g(\nabla_X Y, Z) = X[g(Y, Z)] + Y[g(X, Z)] - Z[g(X, Y)] - g(X, [Y, Z]) - g(Y, [X, Z]) - g(Z, [Y, X]).
\]

Notice that since \( Z \) was arbitrary, and the right hand side depends only on the metric \( g \) and \( Z \), \( \nabla_X Y \) is uniquely determined by the metric. Thus, the Levi-Civita connection is unique.

Now we must show that given an arbitrary Riemannian manifold, \( (M, g) \), there exists a Levi-Civita connection on \( M \). We can define \( \nabla_X Y \) on \( M \) by the Koszul formula, so it remains to show that this definition has the 6 properties of the Levi-Civita connection. Notice that properties 1-4 follow from the linearity properties of the Lie bracket, directional derivative, and the metric. Property 5 follows from computing \( 2g(\nabla_X Y, Z) + 2g(Y, \nabla_X Z) \) and property 6 follows from computing \( 2g(\nabla_X Y - \nabla_Y X, Z) \). □

Let us now consider writing \( \nabla \) as an expression of local coordinates. Let \( x \) be a coordinate patch with coordinates \( x^1, ..., x^n \). Then for vector fields \( X \) and \( Y \) on \( (M, g) \), we can expand

\[
X = \sum_i \phi^i \frac{\partial}{\partial x^i}, \quad \text{and} \quad Y = \sum_j \psi^j \frac{\partial}{\partial x^j}.
\]

So using the linearity properties in the subscript (properties 1 and 2) of the Levi-Civita connection, we can write

\[
\nabla_X Y = \sum_i \phi^i \nabla \frac{\partial}{\partial x^i} \left( \sum_j \psi^j \frac{\partial}{\partial x^j} \right) = \sum_i \phi^i \sum_j \nabla \frac{\partial}{\partial x^i} \psi^j \frac{\partial}{\partial x^j}.
\]

Now we can apply the product rule (property 4) to get

\[
\nabla_X Y = \sum_{ij} \phi^i \psi^j \left( \nabla \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^j} \left[ \psi^j \right] \frac{\partial}{\partial x^i} \right).
\]

To characterize \( \nabla_X Y \) in local coordinates, it is sufficient to calculate \( g(\nabla_X Y, \frac{\partial}{\partial x^k}) \). So from the above expression for \( \nabla_X Y \) and recalling that \( g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g_{ij} \), we find

\[
g(\nabla_X Y, \frac{\partial}{\partial x^k}) = \sum_{ij} \phi^i \psi^j g \left( \nabla \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) + \frac{\partial}{\partial x^j} \left[ \psi^j \right] g_{jk}.
\]
Define \( \Gamma_{ijk} \) by
\[
\Gamma_{ijk} = g \left( \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right).
\]

**Definition (Christoffel Symbols).** Let \((M, g)\) be a Riemannian manifold with Levi-Civita connection \( \nabla \). For a coordinate patch \( x \) with coordinates \( x^1, \ldots, x^n \), the Christoffel symbols of the second kind are
\[
\Gamma_{ijk} = g \left( \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right).
\]
The Christoffel symbols of the first kind are defined by
\[
\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}.
\]
Equivalently, we have
\[
\Gamma_{ijk} = \Gamma^l_{ij} g_{lk} \quad \text{and} \quad \Gamma^k_{ij} = \Gamma_{ijl} g^{lk}.
\]

Notice that by property 4 of the Lie bracket and property 6 of the Levi-Civita connection, we have
\[
\Gamma_{ijk} = g \left( \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) = \Gamma_{jik}
\]
and
\[
\Gamma^k_{ij} = \Gamma^k_{ji}.
\]

In the proof of the uniqueness of the Levi-Civita connection, we established that \( \nabla_X Y \) is completely determined by the metric. We now seek a formula for the Christoffel symbols entirely in terms of the metric \( g \). Observe that
\[
\frac{\partial}{\partial x^k} g_{ij} = \frac{\partial}{\partial x^k} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right).
\]
But by property 5 of the Levi-Civita connection,
\[
\frac{\partial}{\partial x^k} g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = g \left( \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) + g \left( \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) = \Gamma_{kij} + \Gamma_{kji}.
\]
Thus,
\[
\frac{\partial}{\partial x^k} g_{ij} = \Gamma_{ikj} + \Gamma_{jki}.
\]
Similarly, we can derive
\[
\frac{\partial}{\partial x^k} g_{jk} = \Gamma_{ijk} + \Gamma_{kij} \quad \text{and} \quad \frac{\partial}{\partial x^k} g_{ik} = \Gamma_{jki} + \Gamma_{ijk}.
\]
Therefore,
\[
\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij} = \Gamma_{ijk} + \Gamma_{ikj} + \Gamma_{jki} - \Gamma_{ikj} - \Gamma_{jki} = 2 \Gamma_{ijk}.
\]
So we can express \( \Gamma_{ijk} \) entirely in terms of the components of \( g \):
\[
\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij} \right).
\]
Also,
\[
\Gamma^k_{ij} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right) g^{lk}.
\]
For our purposes, the Christoffel symbols of the first kind will be used extensively in future computations. As such, we will simply refer to $\Gamma_{ij}^k$ as the Christoffel symbols for a given manifold.

To give some geometric notion to the Christoffel symbols, let us briefly restrict ourselves to surfaces in $\mathbb{R}^3$. Here, the Levi-Civita connection is simple the covariant derivative defined as the tangential component of the directional derivative. If we have a coordinate patch $x$ with coordinates $x^1$ and $x^2$ and a unit normal $n$ on the surface, we can decompose the second derivative $x_{ij}$ into tangential and normal components,

$$x_{ij} = \Gamma_{ij}^1 \frac{\partial}{\partial x^1} + \Gamma_{ij}^2 \frac{\partial}{\partial x^2} + L_{ij} n,$$

where $\Gamma_{ij}^1$, $\Gamma_{ij}^2$ are indeed Christoffel symbols and $L_{ij}$ are the components of the second fundamental form.

Equation 1.5 is referred to as the Gauss formula for surfaces. More explicitly, the tangential component of the second derivative of $x$ is

$$\text{tang}(x_{ij}) = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} x = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{2} \Gamma_{ij}^k \frac{\partial}{\partial k},$$

Gauss’ formula holds in higher dimensions when considering an $n$-dimensional manifold immersed in $\mathbb{R}^{n+1}$ and can be expressed conveniently using Einstein’s summation notation:

$$x_{ij} = \Gamma_{ij}^k \frac{\partial}{\partial x^k} + L_{ij} n, \quad (\text{Summing over } k = 1, \ldots, n).$$

**Example 6.** Consider the plane $\mathbb{R}^2$ with the Euclidean metric $g = dx^2 + dy^2$ (we will refer to this as the “Euclidean plane”). Then

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \text{and} \quad g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe $\frac{\partial}{\partial x^i} g_{jk} = 0$ for all $i, j, l = 1, 2$. Then by Equation 1.4, $\Gamma_{ij}^k = 0$ for all $i, j, k = 1, 2$.

**Example 7.** Consider the hyperbolic plane $\mathbb{H}^2$ as in Example 5. We previously calculated

$$g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}; \quad \text{and} \quad g^{-1} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}.$$

As a general computation technique, we will calculate $\Gamma_{ij}^k$ one index at a time and use Equation 1.4.

Notice that $g^{ij} = y^2 \delta^i_j$ and $g_{ij} = \frac{1}{y^2} \delta^i_j$. Hence,

$$\Gamma_{ij}^1 = \frac{1}{2} \left( \frac{\partial}{\partial x^i} \frac{1}{y^2} \delta^j_l + \frac{\partial}{\partial x^j} \frac{1}{y^2} \delta^i_l - \frac{\partial}{\partial x^l} \frac{1}{y^2} \delta^i_j \right) y^2 \delta^j_1 \left. \right|_{x^k} \frac{1}{y^2} \delta^l_1 \left. \right|_{x^k},$$

Now we can quickly calculate

$$\Gamma_{11}^1 = \Gamma_{22}^1 = 0, \quad \Gamma_{12}^1 = -\frac{1}{y}.$$

Similarly,

$$\Gamma_{ij}^2 = \frac{1}{2} \left( \frac{\partial}{\partial x^i} \frac{1}{y^2} \delta^j_l + \frac{\partial}{\partial x^j} \frac{1}{y^2} \delta^i_l - \frac{\partial}{\partial x^l} \frac{1}{y^2} \delta^i_j \right) y^2 \delta^j_2 \left. \right|_{x^k} \frac{1}{y^2} \delta^l_2 \left. \right|_{x^k},$$

Thus,

$$\Gamma_{11}^2 = \frac{1}{y}, \quad \Gamma_{12}^2 = 0, \quad \Gamma_{22}^2 = -\frac{1}{y}.$$
Example 8. Consider the sphere as in Example 4. Then
\[
g = \begin{pmatrix} R^2 \sin^2 \phi & 0 \\ 0 & R^2 \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} \frac{1}{R^2} \csc^2 \phi & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix}.
\]

Following the method presented in Example 7, we find
\[
\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{22} = 0
\]
\[
\Gamma^1_{12} = \cot \phi, \quad \Gamma^2_{11} = -\sin \phi \cos \phi.
\]

\section{1.4 Riemann Curvature Tensor}

We now seek a generalization of Gaussian curvature for Riemannian manifolds. For surfaces in \( \mathbb{R}^3 \), Gaussian curvature is the measure of the intrinsic curvature of the surface given by \( K = \lambda_1 \lambda_2 \) where \( \lambda_i \) are the principal curvatures. In a sense, the Gaussian curvature measures how “non-Euclidean” a surface is. Notice that given the Euclidean metric (i.e. flat metric), the corresponding Levi-Civita connection is commutative. The Riemann curvature tensor measures the non-commutativity of the Levi-Civita connection on a manifold. Thus, the curvature tensor measures how non-Euclidean a manifold is. Moreover, the Riemann curvature tensor contains all curvature information of the manifold. The Riemann curvature tensor will allow us to write and prove the Gauss-Codazzi-Mainardi equations and impose conditions on the manifolds that are the result of immersions.

\begin{definition}[Riemann Curvature Tensor] Let \((M, g)\) be a Riemannian manifold. The \textit{Riemann curvature tensor} is the \((1,3)\)-tensor with coefficients
\[
R^l_{ijk} = \frac{\partial}{\partial x^j} \Gamma^l_{ik} - \frac{\partial}{\partial x^k} \Gamma^l_{ij} + \Gamma^l_{ik} \Gamma^p_{pj} - \Gamma^p_{ij} \Gamma^l_{pk}.
\]

Using the Levi-Civita connection, for differentiable vector fields \( X, Y, Z \) on \( M \),
\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]
\end{definition}

Observe that given a coordinate patch with coordinates \( x^1, \ldots, x^n \), since \( \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0 \), the Riemann curvature tensor can be simplified for any differential vector field \( Z \) as
\[
R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) Z = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} Z - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} Z.
\]  \hspace{1cm} (1.7)

This formula emphasizes the above comments that the Riemann curvature tensor measures the non-commutativity of the Levi-Civita connection. Note that from Equation 1.7 and the definition of Christoffel symbols, one can simply compute
\[
R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = -R^l_{kji}.
\]  \hspace{1cm} (1.8)

Moreover, by our convention of raising and lowering indices, and Equation 1.8, we have
\[
g \left( R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right) = -R_{kjl}.
\]  \hspace{1cm} (1.9)
To give more geometrical intuition to the Riemann curvature tensor, we can consider the parallel transport of vectors along loops in our manifold. Without giving the details, if a vector is parallel transported around a loop in Euclidean space, it will point in its original direction at the completion of the loop. For non-Euclidean manifolds, this property does not hold; the Riemann curvature tensor measures the failure of the property. The effect of the curvature on a manifold to parallel transport along loops is called the holonomy of the surface.

Let us now consider the relationship between Gaussian curvature and the Riemann curvature tensor. Gaussian curvature is defined on 2-dimensional surfaces in \( \mathbb{R}^3 \), where there are two principal curvatures and directions. For manifolds of higher dimension, we can consider 2-dimensional sections of the manifold by choosing two linearly independent vectors in the tangent space and looking at the curvature of the plane spanned by those vectors. This curvature is called sectional curvature and can be calculated from the Riemann curvature tensor.

**Definition (Sectional Curvature).** Given a Riemannian manifold \((M, g)\) and linearly independent \(u, v \in T_p M\), the sectional curvature of \(M\) in the plane of \(u\) and \(v\)

\[
K(u, v) = \frac{g(R(u, v)u, v)}{g(u, u)g(v, v) - g(u, v)^2}.
\]

If we consider the sectional curvature of the plane spanned by \(\frac{\partial}{\partial x^i}\) and \(\frac{\partial}{\partial x^j}\), we have

\[
K\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{R_{ijij}}{\det(g_{ii} g_{jj} - g_{ij}^2)},
\]

where \(R_{ijij} = g_{jj} R^s_{ijij}\).

For 2-dimensional manifolds, including surfaces in \(\mathbb{R}^3\), for any two linearly independent vectors in the 2-dimensional tangent space, the sectional curvature agrees with the Gaussian curvature. Moreover, since \(K(u, v)\) does not depend on the choice of \(u\) and \(v\) in the 2-dimensional case, \(K(u, v) \equiv K\) where \(K\) is a constant. Thus, for 2-dimensional manifolds, we can express the Riemann curvature tensor as

\[
R_{1212} = K \left(g_{11} g_{22} - g_{12}^2\right),
\]

where \(K\) is precisely the Gaussian curvature.

Similarly, if \(M\) is an \(n\)-dimensional manifold with constant sectional curvature \((K(u, v) \equiv K)\), we can write the Riemann curvature tensor as

\[
R_{ijkl} = K \left(g_{ik} g_{jl} - g_{il} g_{jk}\right).
\]

In fact, there are only three types of Riemannian manifolds with constant sectional curvature. For sectional curvature \(K = -1\), we say the Riemannian manifold has hyperbolic geometry. \(K = 0\) gives Euclidean geometry, and positive sectional curvature \(K = 1\) gives elliptic geometry. Riemannian manifolds with constant sectional curvature are often called space forms. For more information concerning spaces of constant curvature, see [3, 5].
Example 9. Consider the sphere parametrized in Example 4. By Example 8, we have
\[ \Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{22} = 0 \]
\[ \Gamma^1_{12} = \cot \phi, \quad \Gamma^2_{11} = -\sin \phi \cos \phi. \]

We will calculate the sectional curvature \( K \) and show that \( K \) is precisely the Gaussian curvature of the sphere. Note that we need only calculate \( K = K\left( \partial/\partial \theta, \partial/\partial \phi \right) \) as there are only two tangent directions and the sectional curvature is symmetric. Observe that via lowering indices,
\[ R_{1212} = g_{22} R^s_{121}. \]

Moreover,
\[ R^s_{121} = \frac{\partial}{\partial \phi} \Gamma^s_{11} - \frac{\partial}{\partial \theta} \Gamma^s_{12} + \Gamma^p_{11} \Gamma^s_{p2} - \Gamma^p_{12} \Gamma^s_{p1}. \]

Substituting in the Christoffel symbols, we can calculate
\[ R^s_{121} = 0 \quad \text{and} \quad R^s_{121} = \sin^2 \phi. \]

Recall that
\[ g = \begin{pmatrix} R^2 \sin^2 \phi & 0 \\ 0 & R^2 \end{pmatrix}. \]

So,
\[ R_{1212} = g_{22} R^s_{121} = g_{22} R^s_{121} = R^2 \sin^2 \phi. \]

Thus,
\[ K = \frac{R_{1212}}{\det(g_{11} g_{12} g_{21} g_{22})} = \frac{R^2 \sin^2 \phi}{R^4 \sin^2 \phi} = \frac{1}{R^2}. \]

1.5 Immersions, Embeddings, and Isometries

Up until now, we have been considering Riemannian manifolds as their own object without reference to an “ambient space”. Now we will shift our attention and consider mapping a manifold \((M, g)\) into another manifold \((\bar{M}, \bar{g})\). When \(M\) is immersed in another manifold \(\bar{M}\), we say that \(\bar{M}\) is the ambient space of \(M\). Since the ambient space is also a manifold, we can talk about everything that we have previously done on both the ambient space and our manifold. In order to reduce confusion, we will generally use bars to refer to the ambient space.

If an \(n\)-dimensional manifold \(M\) is immersed in an \((n + k)\)-dimensional manifold \(\bar{M}\), we call \(k\) the co-dimension of \(M\) in \(\bar{M}\). The co-dimension is the number of “normal” directions to \(M\). A familiar example is a surface in \(\mathbb{R}^3\) is 2-dimensional, so the co-dimension is 1. But a curve in \(\mathbb{R}^3\) is 1-dimensional so it has co-dimension 2. This is seen in the Frenet frame where there is one tangential component \(T\) and \(N\) and \(B\) are the normal and bi-normal components, respectively. For our purposes, we will only consider the case where the co-dimension is 1. Examples of co-dimension 1 and 2 in \(\mathbb{R}^3\) are shown in Figure 1.2

There are various ways to map \(M\) into \(\bar{M}\) each preserving different properties. We will consider any map \(F : M \to \bar{M}\) to be differentiable and require that the differential \(dF : T_pM \to T_{p\bar{M}}\) is of full rank. This is equivalent to saying that \(dF\) is injective at each point \(p \in M\). Requiring solely this property gives us the notion of immersions. Further requirements on \(dF\) give additional structure to the image manifold. Immersions, embeddings, and isometries are the most important maps to this discussion and are defined as follows.
**Definition** (Immersion). A differentiable map $F : M \to \bar{M}$ is an **immersion** if $dF$ is of full rank. If there exists such a map, we say $M$ is **immersible** in $\bar{M}$.

**Definition** (Embedding). A differential map $F : M \to \bar{M}$ is an **embedding** if $dF$ is of full rank and $F$ is injective. If there exists such a map, we say $M$ is **embeddable** in $\bar{M}$.

Suppose $M$ is immersed in $\bar{M}$ with co-dimension 1, by immersion $\phi$. At any point $p \in M$, we can talk about both $T_{\phi(p)}\phi(M)$ and $T_{\phi(p)}\bar{M}$, where the difference in dimension is precisely the co-dimension. Moreover, if $X$ and $Y$ are tangent vector fields to $\phi(M)$, then $X,Y$ are also tangent vector fields to $\bar{M}$. Therefore, $\nabla_X Y$ makes sense when $\nabla$ is the Levi-Civita connection on $\bar{M}$ and $X,Y$ are tangent vector fields to $\phi(M)$. Moreover, he metric associated with $\phi(M)$ is the induced metric given by $h = \bar{g}|_{\phi(M)}$, that is, the restriction of the metric on $\bar{M}$ to the immersed manifold $\phi(M)$. When the induced metric agrees with the pull-back metric of $\phi$, we say that $\phi$ is an **isometry**.

**Definition** (Isometry). A differential map $F : M \to \bar{M}$ is an **isometry** if $dF$ preserves the metric $g$. That is, for all $v,w \in T_pM$,

$$\bar{g}(dF(v),dF(w)) = g(v,w).$$

Isometries and local isometries are particularly important in the field of differential geometry since one can define metrics on a manifold via pullback through an isometry. In such a way, if $\bar{M}$ is a Riemannian manifold and $M$ is a set with $\phi : M \to \bar{M}$ a differential map, one can define a metric on $M$ via pullback through $\phi$. Thus, $\phi$ becomes an isometry between $M$ and $\bar{M}$ and $M$ is a Riemannian manifold with respect to the pullback metric. Another important result is Gauss’ Theorema Egregium which states that for surfaces in $\mathbb{R}^3$, Gaussian curvature is an isometric invariance. In other words, if there exists an isometry between surfaces $M$ and $\bar{M}$ in $\mathbb{R}^3$, then $M$ and $\bar{M}$ have the same Gaussian curvature. A quick application of the Theorema Egregium is the inability to create a distance and area preserving map of the Earth on flat paper since the unit sphere has Gaussian curvature 1 (as seen in example 9).
Another area of differential geometry focuses on giving criteria for when an immersed surface is in fact an embedding. That is, what conditions on \( M \) guarantee that if \( M \) can be immersed in \( M \), then \( M \) can be embedded in \( \bar{M} \). These theorems are generally referred to as Hadamard-Stoker type theorems, inspired by \([1, 6]\). The result in chapter 3 can be rephrased as such a theorem, saying that the only isometric immersions of \( \mathbb{R}^2 \) into \( \mathbb{H}^3 \) are horospheres (which are in fact embeddings) and equidistant cylinders (which are only immersions).

Figure 1.3: The difference between immersed and embedded as illustrated by surfaces in \( \mathbb{R}^3 \). The left image is Eneper’s minimal surface and is immersed in \( \mathbb{R}^3 \), while the right image is a helicoid embedded in \( \mathbb{R}^3 \).

1.6 Second Fundamental Form

We previously mentioned via the Gauss formula for surfaces in \( \mathbb{R}^3 \) (Equation 1.6) that we can decompose the second derivative of a coordinate patch \( \mathbf{x} \) as

\[
\mathbf{x}_{ij} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} + L_{ij} \mathbf{n},
\]

where the Christoffel symbols are the tangential components and \( L_{ij} \) are the normal components. In fact, for a surface in \( \mathbb{R}^3 \), we define the second fundamental form to be the tensor with coefficients \( L_{ij} \). Moreover, Gauss’ formula gives us a way to compute the coefficients of the second fundamental form for surfaces by noting that

\[
\mathbf{x}_{ij} \cdot \mathbf{n} = \Gamma^k_{ij} \left( \frac{\partial}{\partial x^k} \cdot \mathbf{n} \right) + L_{ij} (\mathbf{n} \cdot \mathbf{n}) = L_{ij}. \tag{1.11}
\]

Let us use our notion of the Levi-Civita connection and a general metric tensor to rewrite Equation 1.11. If \( M \) is a surface in \( \mathbb{R}^3 \), then the associated Levi-Civita connection \( \nabla \) for the ambient space \( \mathbb{R}^3 \) is given by the Euclidean directional derivative. The metric tensor \( g \) on \( \mathbb{R}^3 \) is given by the usual inner product on \( \mathbb{R}^3 \), \( g(x, y) = x \cdot y \). Thus, we can rewrite Equation 1.11 as

\[
L_{ij} = g \left( \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \mathbf{n} \right). \tag{1.12}
\]

Written in this form, it is easy to generalize the second fundamental form from surfaces to Riemannian manifolds immersed with co-dimension 1.
**Definition** (Second Fundamental Form). Let $M$ be an $n$-dimensional Riemannian manifold. Suppose $M$ is immersed in an $n + 1$-dimensional Riemannian manifold $\bar{M}$ with metric tensor $\bar{g}$ and Levi-Civita connection $\bar{\nabla}$. Let $\mathbf{n}$ be the unit normal to $M$ in $\bar{M}$. Then the Second Fundamental Form is defined by the coefficients

$$L_{ij} = \bar{g} \left( \mathbf{n}, \bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right).$$

For arbitrary vector fields $X$ and $Y$ on $M$, the second fundamental form is given by

$$II(X, Y) = \bar{g} \left( \mathbf{n}, \bar{\nabla}_X Y \right).$$

In classical literature, the second fundamental form is often expressed

$$II(X, Y) = \bar{g} \left( \mathbf{n}, \bar{\nabla}_X Y \right) = \bar{g} \left( \mathbf{n}, -S X \mathbf{n} + \nabla_X \mathbf{n} \right).$$

(1.13)

Here, $S$ is the shape operator of $M$ defined by $S(X) = \nabla_X \mathbf{n}$. The equivalence between this alternate expression and our definition is simple to calculate. Let $X, Y$ be tangent vector fields on the immersed manifold $M$. Then

$$X \left[ \bar{g} \left( \mathbf{n}, Y \right) \right] = X \left[ 0 \right] = 0$$

since $Y$ is tangent and $\mathbf{n}$ is the unit normal. But by property 5 of the Levi-Civita connection, we also have

$$X \left[ \bar{g} \left( \mathbf{n}, Y \right) \right] = \bar{g} \left( \nabla_X \mathbf{n}, Y \right) + \bar{g} \left( \mathbf{n}, \nabla_X Y \right).$$

Combining the two above equations give

$$II(X, Y) = \bar{g} \left( -\nabla_X \mathbf{n}, Y \right) = \bar{g} \left( \mathbf{n}, \nabla_X Y \right).$$

Using Equation 1.13, we will be able to derive the Weingarten equations which we will use to prove the fundamental theorem of surfaces.

Suppose $M$ is immersed in $\bar{M}$ with co-dimension 1. If $\bar{g}$ is the metric on $\bar{M}$ with Levi-Civita connection $\bar{\nabla}$ and second fundamental form $II$, and $\bar{g}$ is the metric on $M$ with Levi-Civita connection $\nabla$, then as a consequence of the definition of the second fundamental form, for any differentiable vector fields $X, Y$ on $M$, we can decompose $\bar{\nabla}_X Y$ into components tangential to $M$ and components normal to $M$ by

$$\nabla_X Y = \nabla_X Y + II(X, Y) \mathbf{n}.$$

(1.14)

This expression recovers the Gauss formula for immersed manifolds. If $M$ is immersed in $\bar{M}$ with co-dimension 1, then

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} + L_{ij} \mathbf{n}.$$

(1.15)

Moreover, if $X = \frac{\partial}{\partial x^k}$ and $Y = \mathbf{n}$, we can express $\nabla_X Y$ using the coefficients of the metric and second fundamental form in the Weingarten equations.
Proposition 4 (Weingarten Equations). Let $M$ be a Riemannian manifold immersed in $\bar{M}$ with codimension 1. Then

$$\bar{\nabla}_{\partial_{x^k}} n = -g^{ij} L_{jk} \frac{\partial}{\partial x^i}.$$ 

Proof. Equation $1.13$ gives us $II(X,Y) = g \left( -\bar{\nabla}_X n, Y \right)$. Observe that since $g(n,n) = 1$,

$$0 = \frac{\partial}{\partial x^i} [g(n,n)] = 2g \left( \bar{\nabla}_{\partial_{x^i}} n, n \right).$$

Thus, $\bar{\nabla}_{\partial_{x^i}} n$ is orthogonal to $n$, so $\bar{\nabla}_{\partial_{x^i}} n \in T_p M$. Since $\{\partial_{x^i}\}$ is a basis for $T_p M$, there exist scalars $C_k^i$ such that

$$\bar{\nabla}_{\partial_{x^k}} n = C_k^i \frac{\partial}{\partial x^i}.$$ 

But,

$$L_{ij} = g \left( -\bar{\nabla}_{\partial_{x^i}} n, \frac{\partial}{\partial x^j} \right) = g \left( -C_k^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = -C_k^i g_{ij}.$$ 

Multiplying by $g^{\alpha \beta}$ gives

$$g^{\alpha \beta} L_{ij} = -C_k^i g^{\alpha \beta} g_{ij} = -C_k^\alpha.$$ 

Therefore,

$$\bar{\nabla}_{\partial_{x^i}} n = -g^{ij} L_{jk} \frac{\partial}{\partial x^i}$$

as desired. \qed

Example 10. Once again, consider the sphere as in Example 4. The parametrization is

$$x(\theta, \phi) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi)$$

for $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$. Then

$$\frac{\partial}{\partial \theta} = ( -R \sin \theta \sin \phi, R \cos \theta \sin \phi, 0)$$

$$\frac{\partial}{\partial \phi} = ( R \cos \theta \cos \phi, R \sin \theta \sin \phi, -R \sin \phi)$$

Our normal $n$ is given by

$$n = \frac{\frac{\partial}{\partial \theta} \times \frac{\partial}{\partial \phi}}{\left\| \frac{\partial}{\partial \theta} \times \frac{\partial}{\partial \phi} \right\|}.$$ 

A quick calculation gives $n = ( -\cos \theta \sin \phi, -\sin \theta \sin \phi, -\cos \phi )$. Furthermore,

$$x_{11} = ( -R \cos \theta \sin \phi, -R \sin \theta \sin \phi, 0)$$

$$x_{12} = ( -R \sin \theta \cos \phi, R \cos \theta \cos \phi, 0)$$

$$x_{22} = ( -R \cos \theta \sin \phi, -R \sin \theta \sin \phi, -R \cos \phi).$$

Now we can calculate the coefficients of the second fundamental form $II_{ij} = x_{ij} \cdot n$:

$$II_{11} = R \sin^2 \phi; \quad II_{12} = 0; \quad II_{22} = R.$$
1.7 Gauss and Codazzi-Mainardi Equations

One of the main tools that we will use in determining how manifolds can be immersed are the Gauss and Codazzi-Mainardi equations. These equations relate the second fundamental form and Christoffel symbols to the Riemann curvature tensor of both the manifold and the ambient space. These equations for surfaces in $\mathbb{R}^3$ also are used in the proof of Gauss’ Theorema Egregium. Here, we will first prove the Gauss and Codazzi-Mainardi equations for surfaces in $\mathbb{R}^3$ then seek to generalize that proof for arbitrary Riemannian manifolds and ambient spaces with co-dimension 1.

**Theorem 5** (Gauss and Codazzi-Mainardi Equations for Surfaces). Let $M$ be a surface in $\mathbb{R}^3$ with metric tensor coefficients $g_{ij}$ and second fundamental form coefficients $L_{ij}$. Then

\begin{align}
\text{(Gauss' Equations)} \quad & R^l_{ijk} = L_{ik}L_{jp}g^{pl} - L_{ij}L_{kp}g^{pl}, \\
\text{(Codazzi-Mainardi Equations)} \quad & \frac{\partial}{\partial x^k}L_{ij} - \frac{\partial}{\partial x^j}L_{ik} = \Gamma^l_{ik}L_{lj} - \Gamma^l_{ij}L_{lk}.
\end{align}

**Proof.** Consider the Gauss formula (Equation 1.6),

$$x_{ij} = \Gamma^l_{ij} \frac{\partial}{\partial x^k} + L_{ij} n.$$  

Differentiating with respect to $x_k$ and substituting Gauss formula and the Weingarten equations (prop. 4), we can write

$$x_{ijk} = \frac{\partial}{\partial x^k}x_{ij} + \frac{\partial L_{ij}}{\partial x^k}n + L_{ij} n_k$$  

$$= \frac{\partial}{\partial x^k}x_{ij} + \Gamma^l_{ij} (\Gamma^p_{lk} x_l + L_{lk} n) + \frac{\partial L_{ij}}{\partial x^k}n + L_{ij} (-L_{pk}g^{pl} x_l)$$  

$$= \frac{\partial}{\partial x^k}x_{ij} + \Gamma^l_{ij} \Gamma^p_{lk} x_l - L_{ij} L_{pk} g^{pl} x_l + \frac{\partial L_{ij}}{\partial x^k}n + \Gamma^l_{ij} L_{lk} n$$  

$$= \left( \frac{\partial}{\partial x^k} + \Gamma^l_{ij} \Gamma^p_{lk} - L_{ij} L_{pk} g^{pl} \right) x_l + \left( \frac{\partial L_{ij}}{\partial x^k} + \Gamma^l_{ij} L_{lk} \right) n.$$  

By the same argument (or relabeling indices), we have

$$x_{ikj} = \left( \frac{\partial}{\partial x^k} + \Gamma^l_{ik} \Gamma^p_{pj} - L_{ik} L_{pj} g^{pl} \right) x_l + \left( \frac{\partial L_{ik}}{\partial x^k} + \Gamma^l_{ik} L_{lj} \right) n.$$  

Since we are considering Euclidean differentiation, $x_{ijk} = x_{ikj}$, so both the tangential and normal components of $x_{ijk} - x_{ikj}$ are zero. The tangential component gives Gauss’ Equations:

$$0 = \frac{\partial}{\partial x^k} + \Gamma^l_{ij} \Gamma^p_{pk} - L_{ij} L_{pk} g^{pl} - \frac{\partial}{\partial x^l} \Gamma^p_{ik} \Gamma^l_{pj} - L_{ik} L_{pj} g^{pl} = \Gamma^l_{ik} L_{lj} - \Gamma^l_{ij} L_{lk}.$$  

The normal component recovers the Codazzi-Mainardi Equations:

$$0 = \frac{\partial}{\partial x^k} + \Gamma^l_{ij} L_{ik} - \frac{\partial L_{ik}}{\partial x^j} - \Gamma^l_{ik} L_{lj}. \quad \square$$

When considering general Riemannian manifolds and ambient spaces, we must take into account the curvature of the ambient space. Note that in the following equations, if we take $\bar{M} = \mathbb{R}^n$, the ambient space curvature terms are zero, so we recover Theorem 5.
Theorem 6 (Gauss and Codazzi-Mainardi Equations for Immersed Manifolds). Let $M$ be an immersed manifold in $\bar{M}$ of co-dimension 1 with metric tensor coefficients $g_{ij}$ and second fundamental form coefficients $L_{ij}$. Let $R$ and $\bar{R}$ be the Riemann curvature tensors of $M$ and $\bar{M}$, respectively. Let $n = \frac{\partial}{\partial \bar{x}^{n}}$ be the vector in $\bar{M}$ that is normal to $M$. Then

\begin{align*}
\text{(Gauss' Equations)} & \quad R_{ijkl} - \bar{R}_{ijkl} = L_{ik}L_{kj} - L_{ij}L_{kl}, \\
\text{(Codazzi-Mainardi Equations)} & \quad \bar{R}_{ijkn} = \frac{\partial}{\partial x^{j}}L_{ik} - \frac{\partial}{\partial x^{k}}L_{ij} + \Gamma_{ik}^{s}L_{js} - \Gamma_{ij}^{s}L_{sk}.
\end{align*}

Proof. We will start with the Gauss formula for an immersed Riemann manifold with co-dimension 1 (Equation 1.15),

$$\nabla_{\frac{\partial}{\partial x^{i}}} \frac{\partial}{\partial x^{j}} = \Gamma_{ij}^{k} \frac{\partial}{\partial x^{k}} + L_{ij}n.
$$

Notice that by property 4 of the Levi-Civita connection for any scalar function $f$ on $M$ and differentiable vector field $X$, $\nabla_{X} f = X[f]$. Thus,

$$\nabla_{\frac{\partial}{\partial x^{i}}} \Gamma_{ij}^{p} = \frac{\partial \Gamma_{ij}^{p}}{\partial x^{k}} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial x^{i}}} L_{ij} = \frac{\partial L_{ij}}{\partial x^{k}}.
$$

Then with substitutions from the Gauss formula and the Weingarten equations (Proposition 4), we can write

$$\nabla_{\frac{\partial}{\partial x^{i}}} \nabla_{\frac{\partial}{\partial x^{j}}} \frac{\partial}{\partial x^{i}} = \nabla_{\frac{\partial}{\partial x^{i}}} \left( \Gamma_{ij}^{p} \frac{\partial}{\partial x^{p}} + L_{ij}n \right)
\begin{align*}
&= \frac{\partial \Gamma_{ij}^{p}}{\partial x^{k}} \frac{\partial}{\partial x^{k}} + \Gamma_{ij}^{p} \nabla_{\frac{\partial}{\partial x^{j}}} \frac{\partial}{\partial x^{p}} + \frac{\partial L_{ij}}{\partial x^{k}} n + L_{ij} \nabla_{\frac{\partial}{\partial x^{i}}} \frac{\partial}{\partial x^{k}} n \\
&= \frac{\partial \Gamma_{ij}^{p}}{\partial x^{k}} \frac{\partial}{\partial x^{k}} + \Gamma_{ij}^{p} \left( \Gamma_{kp}^{s} \frac{\partial}{\partial x^{s}} + L_{kp}n \right) + \frac{\partial L_{ij}}{\partial x^{k}} n + L_{ij} \left( -L_{sk}g^{sp} \frac{\partial}{\partial x^{p}} \right) \\
&= \left( \frac{\partial \Gamma_{ij}^{p}}{\partial x^{k}} + \Gamma_{ij}^{s} \Gamma_{kp}^{s} - L_{ij} L_{sk} g^{sp} \right) \frac{\partial}{\partial x^{p}} + \left( \frac{\partial L_{ik}}{\partial x^{j}} + \Gamma_{ik}^{p} L_{jp} \right) n.
\end{align*}
$$

Similarly, we can interchange $k$ and $j$ to get

$$\nabla_{\frac{\partial}{\partial x^{j}}} \nabla_{\frac{\partial}{\partial x^{k}}} \frac{\partial}{\partial x^{i}} = \left( \frac{\partial \Gamma_{ij}^{p}}{\partial x^{k}} + \Gamma_{ij}^{s} \Gamma_{kp}^{s} - L_{ij} L_{sk} g^{sp} \right) \frac{\partial}{\partial x^{p}} + \left( \frac{\partial L_{ik}}{\partial x^{j}} + \Gamma_{ik}^{p} L_{jp} \right) n.
$$

By Equation 1.7, we have

$$\bar{R} \left( \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{j}} \right) \frac{\partial}{\partial x^{l}} = \nabla_{\frac{\partial}{\partial x^{k}}} \nabla_{\frac{\partial}{\partial x^{l}}} \frac{\partial}{\partial x^{i}} - \nabla_{\frac{\partial}{\partial x^{l}}} \nabla_{\frac{\partial}{\partial x^{k}}} \frac{\partial}{\partial x^{i}}.
$$

In particular by our comments for Equation 1.9, we have that

$$g \left( \nabla_{\frac{\partial}{\partial x^{i}}} \nabla_{\frac{\partial}{\partial x^{j}}} - \nabla_{\frac{\partial}{\partial x^{j}}} \nabla_{\frac{\partial}{\partial x^{i}}} \right) = g \left( \bar{R} \left( \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{j}} \right) \frac{\partial}{\partial x^{l}}, \frac{\partial}{\partial x^{l}} \right) = -\bar{R}_{ijkl}$$

and the normal component

$$\left( \nabla_{\frac{\partial}{\partial x^{i}}} \nabla_{\frac{\partial}{\partial x^{j}}} - \nabla_{\frac{\partial}{\partial x^{j}}} \nabla_{\frac{\partial}{\partial x^{i}}} \right) \perp = g \left( \bar{R} \left( \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{j}} \right) \frac{\partial}{\partial x^{l}}, n \right) = -\bar{R}_{ijkn}.$$
But we previously calculated the tangential and normal components above, so we have

\[-\bar{R}_{ijkl} = g \left( \nabla \frac{\partial}{\partial x^i} \nabla \frac{\partial}{\partial x^j} - \nabla \frac{\partial}{\partial x^j} \nabla \frac{\partial}{\partial x^i} \right) \]

\[= g \left( \left( \partial \Gamma^p_{ij} + \Gamma^p_{kji} - L_{ij} L_{sk} g^{sp} - \frac{\partial \Gamma^p_{ik}}{\partial x^j} - \Gamma^p_{ik} \Gamma^p_{js} + L_{ik} L_{sj} g^{sp} \right) \frac{\partial}{\partial x^p} , \frac{\partial}{\partial x^l} \right) \]

\[= -R_{ijk} - L_{ij} L_{sk} g^{sp} + L_{ik} L_{sj} g^{sp} \frac{\partial}{\partial x^p} , \frac{\partial}{\partial x^l} \]

\[= -R_{ijkl} - L_{ij} L_{lk} + L_{ik} L_{ij} \]

and

\[-\bar{R}_{ijkn} = \left( \nabla \frac{\partial}{\partial x^i} \nabla \frac{\partial}{\partial x^j} - \nabla \frac{\partial}{\partial x^j} \nabla \frac{\partial}{\partial x^i} \right)^\perp \]

\[= \frac{\partial L_{ij}}{\partial x^k} + \Gamma^p_{ij} L_{kp} - \frac{\partial L_{ik}}{\partial x^j} - \Gamma^p_{ik} L_{jp} \]

as desired. \qed
1.8 Fundamental Theorem of Surfaces

We are now able to state and prove the main theorem that will allow us to classify immersed surfaces. The fundamental theorem of surfaces was proved by Pierre Bonnet in 1860 and states that the first and second fundamental forms uniquely determines surfaces up to a rigid motion. When classifying immersed surfaces, we can use the Gauss and Codazzi-Mainardi equations to provide constraints to the first and second fundamental forms of the immersed surface. If we can determine and identify the first and second fundamental forms as equivalent to other known surfaces, then the fundamental theorem of surfaces tells us that the immersed surface is the known surface up to a Euclidean motion. For the Volkov and Vladimirova result in Chapter 3, we will use a more general theorem for immersed manifolds. Essentially, the fundamental theorem of surfaces holds locally for arbitrary manifolds and the result holds globally when the manifold is simply connected. The general version of the fundamental theorem of surfaces is called the Bonnet theorem and is presented in [5].

Theorem 7 (Fundamental Theorem of Surfaces). Let $U \subseteq \mathbb{R}^2$ be a convex open set containing $(0,0)$.

1. Let $M, M' \subseteq \mathbb{R}^3$ be two immersed surfaces with coordinate patches $\mathbf{x}, \bar{\mathbf{x}} : U \to \mathbb{R}^3$, respectively. Let $g_{ij}, L_{ij}$ be the coefficients of the first and second fundamental form on $M$ and $\bar{g}_{ij}, L_{ij}$ be the coefficients of the first and second fundamental form on $M'$ under the coordinate patches $\mathbf{x}$ and $\bar{\mathbf{x}}$. If $g_{ij} = \bar{g}_{ij}$ and $L_{ij} = \bar{L}_{ij}$ on $U$, then $M$ and $M'$ are equivalent up to a proper Euclidean motion. That is, there is a proper Euclidean motion $A$ such that $\bar{\mathbf{x}} = A \circ \mathbf{x}$.

2. Suppose $g_{ij}$ and $L_{ij}$ be functions on $U$ which satisfy:
   (a) $g_{ij} = g_{ji}$ and $L_{ij} = L_{ji}$, and $(g_{ij})$ is positive definite on $U$, so that we can define the corresponding $g^{ij}$ and $\Gamma^k_{ij}$.
   (b) Gauss' equation:
      $$L_{11}L_{22} - L_{12}^2 = R_{1212} = g_{1k} \left( \frac{\partial}{\partial x^1} \Gamma^k_{22} - \frac{\partial}{\partial x^2} \Gamma^k_{21} - \Gamma^p_{22} \Gamma^k_{p1} - \Gamma^p_{21} \Gamma^k_{p2} \right) ,$$
   (c) The Codazzi-Mainardi equations:
      $$\frac{\partial}{\partial x^1} L_{12} - \frac{\partial}{\partial x^2} L_{11} + \Gamma^k_{12} L_{k1} - \Gamma^k_{11} L_{k2} = 0$$
      and
      $$\frac{\partial}{\partial x^1} L_{22} - \frac{\partial}{\partial x^2} L_{21} + \Gamma^k_{22} L_{k1} - \Gamma^k_{21} L_{k2} = 0 .$$
   Then there is an immersed surface $M \subseteq \mathbb{R}^3$ with coordinate patch $\mathbf{x} : U \to \mathbb{R}^3$ whose coefficients of the first and second fundamental form are precisely $g_{ij}$ and $L_{ij}$, respectively.

Proof. 1. Suppose $M, M' \subseteq \mathbb{R}^3$ are positively oriented. That is,

$$N = \frac{\frac{\partial}{\partial x^1} \times \frac{\partial}{\partial x^2}}{\sqrt{g_{11}g_{22} - (g_{12})^2}}$$

and similarly for $\bar{N}$. For ease of reference, let us adopt the following notation:

$$v_1 = \frac{\partial}{\partial x^1}, \quad v_2 = \frac{\partial}{\partial x^2}, \quad v_3 = N$$

and

$$\bar{v}_1 = \frac{\partial}{\partial \bar{x}^1}, \quad \bar{v}_2 = \frac{\partial}{\partial \bar{x}^2}, \quad \bar{v}_3 = \bar{N}.$$
Since $g_{ij}(0) = \tilde{g}_{ij}(0)$ and $(v_1(0,0), v_2(0,0), v_3(0,0))$, $(\tilde{v}_1(0,0), \tilde{v}_2(0,0), \tilde{v}_3(0,0))$ are both positively oriented by hypotheses, there exists a rotation $B \in SO(3)$ such that

$$B(v_i(0,0)) = \tilde{v}_i(0,0) \quad \text{for } i = 1, 2, 3.$$  

Define $\mathbf{x} = B \circ \mathbf{x}$ and let $\hat{g}_{ij}$ be the coefficients of the first fundamental form, $\hat{L}_{ij}$ be the coefficients of the second fundamental form, and $\hat{v}_i$ be defined as above for $\mathbf{x}$. Since $B$ is an isometry, we have

$$\hat{g}_{ij} = g_{ij} = \tilde{g}_{ij}$$

$$\hat{v}_3 = B \circ v_3$$

$$\hat{L}_{ij} = L_{ij} = \tilde{L}_{ij}.$$  

Moreover, as constructed, $(\tilde{v}_1(0,0), \tilde{v}_2(0,0), \tilde{v}_3(0,0)) = (\tilde{v}_1(0,0), \tilde{v}_2(0,0), \tilde{v}_3(0,0))$. We will show that $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ on all of $U$. Consider the Gauss formula (Equation 1.6) and the Weingarten equations (Proposition 4) on $\mathbf{x}$:

$$\begin{align*}
\frac{\partial}{\partial x^i} v_1(s, t) &= \Gamma^h_{ik}(s, t) v_k(s, t) + L_{ik}(s, t) v_3(s, t) \quad \text{for } i = 1, 2 \\
\frac{\partial}{\partial x^i} v_3(s, t) &= -g^{kj}(s, t) L_{kj}(s, t) v_h(s, t)
\end{align*}$$

Similarly, we have the same equations for $\mathbf{x}$ with $\hat{g}_{ij}$, $\hat{L}_{ij}$, and $\hat{\Gamma}_{ij}$. But $\hat{L}_{ij} = \tilde{L}_{ij}$ and since $\hat{g}_{ij} = \tilde{g}_{ij}$, we also have $\hat{\Gamma}_{ij} = \tilde{\Gamma}_{ij}$ and $\hat{\Gamma}_{ij} = \tilde{\Gamma}_{ij}$. Thus, both $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ and $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ satisfy Equation 1.20 and take the same values at $(0, 0)$. Thus, by the existence and uniqueness theorem of nonlinear systems of PDEs, $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ on $U$. Thus, $\mathbf{x}$ and $\tilde{\mathbf{x}}$ have the same partial derivatives so they must differ by only a constant vector. Therefore, there exists a translation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathbf{x} = T \circ \mathbf{x} = T \circ B \circ \mathbf{x}$ as desired.

2. Suppose $g_{ij}$ and $L_{ij}$ are functions on $U$ that satisfy properties (a)-(c). Our goal is to show the existence of $\mathbf{x} : U \to \mathbb{R}^3$ whose coefficients of the first and second fundamental form agree with $g_{ij}$ and $L_{ij}$.

Consider the system of PDE’s given in Equation 1.20. By the theorem of uniqueness and existence for a system of PDE’s and since $g_{ij}$ and $L_{ij}$ satisfy the Gauss and Codazzi-Mainardi equations, 1.20 has a unique solution $v_1, v_2, v_3 : U \to \mathbb{R}^3$ given any initial conditions. Moreover, since Equation 1.20 is a linear system, the solution is globally defined on $U$. By the hypothesis that $(g_{ij})$ is positive definite, we can find the unique solution to 1.20 given the following initial conditions as $(0, 0)$:

(a) $g(v_i(0,0), v_j(0,0)) = g_{ij}(0,0) \quad \text{for } i, j = 1, 2$

(b) $g(v_i(0,0), v_3(0,0)) = 0 \quad \text{for } i = 1, 2$

(c) $||v_3(0,0)|| = 1$

(d) $(v_1(0,0), v_2(0,0), v_3(0,0))$ is positively oriented.

We will now show that (a)-(d) are satisfied on all of $U$. Since $v_i$ are solutions to Equation 1.20, the following equations hold:

$$\frac{\partial}{\partial x^k}[g(v_i, v_j)] = g \left( \frac{\partial}{\partial x^k} v_i, v_j \right) + g(v_i, \frac{\partial}{\partial x^k} v_j)$$

$$= \Gamma^h_{ik} g(v_h, v_j) + \Gamma^h_{jk} g(v_h, v_i) + L_{ik} g(v_3, v_i) + L_{jk} g(v_3, v_j)$$

$$\frac{\partial}{\partial x^k}[g(v_i, v_3)] = L_{ik} - g^{kj} L_{kj} g(v_i, v_h)$$

$$\frac{\partial}{\partial x^k}[g(v_3, v_3)] = 2g \left( \frac{\partial}{\partial x^k} v_3, v_3 \right)$$
But, by differentiating \( g_{ij} \) with respect to \( x^k \), we have

\[
\frac{\partial}{\partial x^k} g_{ij} = \frac{\partial}{\partial x^k} \left[ g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right] = g \left( \nabla \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) + g \left( \frac{\partial}{\partial x^i}, \nabla \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^j} \right) = \Gamma^h_{ik} g_{hj} + \Gamma^h_{jk} g_{hi}.
\]

(1.24)

Note that in the above equations, 1.21 through 1.24, \( i, j, h = 1, 2 \). Observe that Equation 1.24 shows that equations 1.21 through 1.23 hold for \( g(v_i, v_j) = g_{ij}, g(v_3, v_i) = 0, \) and \( g(v_3, v_3) = 1 \). Since (a)-(c) guarantee that the above substitutions are equivalent at \((0,0)\), it follows that the substitutions agree on all of \( U \). Hence, (a)-(c) are satisfied on all of \( U \). Then (a) and (b) guarantee that \((v_1, v_2, v_3)\) are linearly independent, so if \((v_1, v_2, v_3)\) are positively oriented at \((0,0)\), then \((v_1, v_2, v_3)\) are positively oriented on the entirety of \( U \). Thus (d) holds on all of \( U \).

It remains to show that there exists a function \( \mathbf{x} : U \to \mathbb{R}^3 \) with \( \frac{\partial \mathbf{x}}{\partial x^i} = v_i \) for \( i = 1, 2 \). It suffices to show that \((v_1, v_2)\) is a conservative vector field on \( U \); therefore, there exists a potential function \( \mathbf{x} \) such that \( \frac{\partial \mathbf{x}}{\partial x^i} = v_i \). Since \( U \) is a convex open set, it is a simply connected open set. Thus, \((v_1, v_2)\) is a conservative vector field if \( \frac{\partial}{\partial x^j} v_i = \frac{\partial}{\partial x^i} v_j \). But this property follows from Equation 1.20 since \( \Gamma^k_{ij} \) and \( g_{ij} \) are symmetric.

Thus, \( g \left( \frac{\partial \mathbf{x}}{\partial x^i}, \frac{\partial \mathbf{x}}{\partial x^j} \right) = g_{ij} \) by (a) and \( v_3 = \mathbf{n} \) by (b)-(d). Therefore, \( g \left( \mathbf{n}, \nabla \frac{\partial \mathbf{x}}{\partial x^j} \right) = L_{ij} \) by Equation 1.20 with (b) and (d). We have thus constructed \( \mathbf{x} : U \to \mathbb{R}^3 \) whose coefficients of the first and second fundamental form agree with \( g_{ij} \) and \( L_{ij} \).

□

24
Chapter 2

Hyperbolic 3-Space

2.1 $\mathbb{H}^3$ as a Subset of Minkowski Spacetime

Consider $\mathbb{R}^4$ with the vector notation $(t, x) = (t, x_1, x_2, x_3)$. $\mathbb{R}^4$ endowed with the metric

$$ h = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 $$

is called Minkowski spacetime and will be denoted as $\mathbb{L}^4$. Note that $h$ as defined does not satisfy the properties of a Riemannian metric since it is not positive definite. We say that Minkowski spacetime is a Pseudo-Riemannian manifold because $h$ is not positive definite. More specifically, Minkowski spacetime is called a Lorentzian manifold and $h$ is a Lorentzian metric since the $t$ dimension has an opposite time to the other dimensions.

Intuitively, $\mathbb{L}^4$ can be thought of as $t$ being “time” and $x$ being “space”. We can alternatively write the metric as

$$ h = -dt^2 + |dx|^2 $$

where $|dx|^2$ is the Euclidean metric on $\mathbb{R}^3$. The structure of the metric allows us to label vectors as time-like, space-like, and light-like. A vector $v \in \mathbb{L}^4$ is time-like if $h(v, v) < 0$, space-like if $h(v, v) > 0$, and light-like or null if $h(v, v) = 0$. Moreover, we can consider the hyperquadrics

$$ \mathbb{H}^3 = \{(t, x) \in \mathbb{L}^4 | -t^2 + |x|^2 = -1; t > 0\} $$
$$ \mathbb{S}_1^3 = \{(t, x) \in \mathbb{L}^4 | -t^2 + |x|^2 = 1\} $$
$$ \mathbb{N}_+^3 = \{(t, x) \in \mathbb{L}^4 | -t^2 + |x|^2 = 0; t > 0\}, $$

where $\mathbb{H}^3$ is hyperbolic 3-space, $\mathbb{S}_1^3$ is de Sitter space, and $\mathbb{N}_+^3$ is the positive null cone. A visualization of $\mathbb{H}^3$ and $\mathbb{N}_+^3$ is given in Figure 2.1.

If we consider $\mathbb{H}^3$ as a manifold, it inherits the metric from $\mathbb{L}^4$ by restriction:

$$ g := h \bigg|_{\mathbb{H}^3} = -dt^2 + |dx|^2 \bigg|_{\mathbb{H}^3}. $$

Importantly, all of the results from Chapter 1 hold for Pseudo-Riemannian manifolds, so we can still talk of the Levi-Civita connection, Christoffel symbols, curvature tensor, etc on $\mathbb{H}^3$ when defined as a subset of $\mathbb{L}^4$. 
2.2 Poincaré Ball Model

The Poincaré ball model for hyperbolic 3-space arises via stereographic projection in \( L^4 \). In Minkowski spacetime, the hyperplane with \( t = 0 \) is isomorphic to \( \mathbb{R}^3 \), and thus contains an isomorphic copy of \( B^3 \). Explicitly, this is the set \( B^3_L = \{(0, x) \in L^3 | |x|^2 \leq 1\} \). Notice that \( \mathbb{H}^3 \) is defined for \( t > 0 \). If we instead consider the lower half of the hyperboloid, we have that the lower vertex is \((-1, 0, 0, 0)\). Then stereographic projection from the lower vertex through a point \((0, y) \in B^3_L \) to \((t, x) \in \mathbb{H}^3 \) defines a diffeomorphism between \( B^3_L \) and \( \mathbb{H}^3 \). The stereographic projection map is depicted in Figure 2.2.

We can calculate the map directly. The line from the lower vertex \((-1, 0, 0, 0)\) to \((0, y) \in B^3_L \) is given by \( l(s) = (0, y) + s(1, y) = (s, (1 + s)y) \).

To solve for \( s \) such that \( l(s) \in \mathbb{H}^3 \), note that

\[
(s, (1 + s)y) \in \mathbb{H}^3 \iff -s^2 + |(1 + s)y|^2 = -1.
\]

Since \((1 + s)\) is a scalar, we can factor it out from the Euclidean metric to get

\[
|y|^2 = \frac{s^2 - 1}{(s + 1)^2} = \frac{s - 1}{s + 1} = 1 - \frac{2}{s + 1}.
\]

Then

\[
s + 1 = \frac{2}{1 - |y|^2} \implies s = \frac{1 + |y|^2}{1 - |y|^2}.
\]

Thus, our mapping \( \phi : B^3 \to \mathbb{H}^3 \) is given by

\[
\phi(y) = \left( \frac{1 + |y|^2}{1 - |y|^2}, \frac{2}{1 - |y|^2}, y \right).
\]

This stereographic projection is an isometry between \( B^3 \) and \( \mathbb{H}^3 \) under the pull-back metric, where \( \partial B^3 = S^2 \) is the ideal boundary of hyperbolic space at infinity. Let us calculate the pullback metric \( \phi^* g \).
Recall that the pullback metric is defined to be
\[ \phi^* g(u, v) = g(d\phi(u), d\phi(v)). \]
Since \( g = -dt^2 + \sum_{i=1}^3 dx_i^2 \), we will calculate \( dt^2 \) and \( dx_i^2 \) for the explicit formula that we have from \( \phi \). Since \( t = \frac{1+|y|^2}{1-|y|^2} \), let us first calculate \( d|y|^2 \).

Thus,
\[
d|y|^2 = \sum_{i=1}^3 \left( \frac{\partial}{\partial y_i} |y|^2 \right) dy_i = \sum_{i=1}^3 \left( \frac{\partial}{\partial y_i} \sum_{j=1}^3 y_i^2 \right) dy_i
\]
\[= \sum_{i,j=1}^3 2y_j \delta_{ij} dy_i = \sum_{i=1}^3 2y_i dy_i.
\]

Then,
\[
dt^2 = \left( \frac{(1-|y|^2)dt^2 + (1+|y|^2)d|y|^2}{(1-|y|^2)^2} \right)^2
\]
\[= \left( \frac{2}{1-|y|^2} \sum_{i=1}^3 y_i^2 \right)^2
\]
\[= \frac{2}{1-|y|^2} \sum_{i=1}^3 y_i^2 \left( \sum_{j=1}^3 y_j dy_j \right)^2.
\]

Each \( x_i = \frac{2y_i}{1-|y|^2} \), so
\[
dx_i = \frac{2}{1-|y|^2} dy_i + y_i \frac{2d|y|^2}{(1-|y|^2)^2}
\]
\[= \frac{2}{1-|y|^2} dy_i + \left( \frac{2}{1-|y|^2} \right)^2 y_i \left( \sum_{j=1}^3 y_j dy_j \right).
\]

Then
\[
dx_i^2 = \left( \frac{2}{1-|y|^2} \right)^2 dy_i^2 + \left( \frac{2}{1-|y|^2} \right)^3 y_i^2 \left( \sum_{j=1}^3 y_j dy_j \right) + \left( \frac{2}{1-|y|^2} \right)^4 y_i^2 \left( \sum_{j=1}^3 y_j dy_j \right)^2.
\]
Now, we can calculate $\phi^*g$ using our substitutions found above.

$$\phi^*g = -dt^2 + \sum_{i=1}^{3} dx_i^2$$

$$= -\left(\frac{2}{1-|y|^2}\right)^4 \left(\sum_{i=1}^{3} y_i dy_i\right)^2 + \left(\frac{2}{1-|y|^2}\right)^2 \sum_{i=1}^{3} dy_i^2$$

$$+ 2 \left(\frac{2}{1-|y|^2}\right)^3 \left(\sum_{i=1}^{3} y_i dy_i\right) \left(\sum_{j=1}^{3} y_j dy_j\right) + \left(\frac{2}{1-|y|^2}\right)^4 \left(\sum_{i=1}^{3} y_i^2\right) \left(\sum_{j=1}^{3} y_j dy_j\right)^2$$

$$= \left(\frac{2}{1-|y|^2}\right)^4 \left(\sum_{j=1}^{3} y_j dy_j\right)^2 (|y|^2 - 1) + \left(\frac{2}{1-|y|^2}\right)^2 \left(\sum_{j=1}^{3} y_j dy_j\right)^2 (1-|y|^2) + \left(\frac{2}{1-|y|^2}\right)^2 |dy|^2$$

$$= \left(\frac{2}{1-|y|^2}\right)^2 |dy|^2.$$

Therefore,

$$\phi^*g = \left(\frac{2}{1-|y|^2}\right)^2 |dy|^2.$$

It is worth pointing out that the pullback metric on $B^3$ is conformal to the Euclidean metric. We have thus shown

$$(\mathbb{H}^3, g) \cong \left(B^3, \left(\frac{2}{1-|y|^2}\right)^2 |dy|^2\right).$$

The Poincaré ball model is much more intuitive that the model of $\mathbb{H}^3$ in Minkowski spacetime, and will allow us to visualize immersed surfaces better than the 4-dimensional counterpart in spacetime. However, computations with the ball model are more tedious due to the complexity of the metric. We will briefly discuss one more model of hyperbolic 3-space which will be convenient to make calculations.

### 2.3 Upper Half Space Model

Recall that in example 5, we modeled hyperbolic 2-space on the upper half plane of $\mathbb{R}^2$. We can generalize that idea for hyperbolic 3-space. Let $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 > 0\}$. If we endow $\mathbb{R}^3_+$ with the metric $\frac{|dx|^2}{x_3^2}$, we have the upper half space model of hyperbolic 3-space. That is,

$$(\mathbb{H}^3, g) \cong \left(\mathbb{R}^3_+, \frac{|dx|^2}{x_3^2}\right),$$

where $|dx|^2$ is the Euclidean metric on $\mathbb{R}^3$.

The mapping between the Poincaré ball model and the upper half space model is given by spherical inversion about a sphere with radius 2 whose center is a boundary point of $B^3$. If we invert about the sphere centered at $(0,0,-1)$, we have the map

$$\varphi : \left(B^3, \left(\frac{2}{1-|y|^2}\right)^2 |dy|^2\right) \to \left(\mathbb{R}^3_+, \frac{|dx|^2}{x_3^2}\right)$$

given by

$$\varphi(x_1, x_2, x_3) = \left(\frac{4x_1}{x_1^2 + x_2^2 + (x_3 + 1)^2}, \frac{4x_2}{x_1^2 + x_2^2 + (x_3 + 1)^2}, \frac{4(x_3 + 1)}{x_1^2 + x_2^2 + (x_3 + 1)^2} - 1\right).$$

28
In fact, \( \varphi \) is a conformal mapping and the boundary of the Poincaré ball is mapped to \( \{ x \in \mathbb{R}^3 | x_3 = 0 \} \cup \{ \infty \} \).

The image of a horosphere whose point at infinity is \((0, 0, -1)\) is a plane under \( \varphi \) and is shown in Figure 2.3.

We will mainly use this model of hyperbolic 3-space to make computations since the metric is a simple conformal metric to the Euclidean metric.

### 2.4 Geometry of \( \mathbb{H}^3 \)

We can now show that the sectional curvature of \( \mathbb{H}^3 \) is constant \(-1\). Consider the upper half space model \( (\mathbb{R}^3_+, \frac{dx^2}{x_3^2}) \). We can write the components of the metric as

\[
g = \begin{pmatrix}
\frac{1}{x_3^2} & 0 & 0 \\
0 & \frac{1}{x_3^2} & 0 \\
0 & 0 & \frac{1}{x_3^2}
\end{pmatrix}
\quad \text{and} \quad
\mathbf{g}^{-1} = \begin{pmatrix}
x_3^2 & 0 & 0 \\
0 & x_3^2 & 0 \\
0 & 0 & x_3^2
\end{pmatrix}.
\]

Using Equation 1.4 to calculate the Christoffel symbols, we have

\[
\Gamma^1_{13} = \Gamma^2_{23} = \Gamma^3_{33} = -\frac{1}{x_3} \quad \Gamma^3_{11} = \Gamma^3_{22} = \frac{1}{x_3}
\]

and all other Christoffel symbols are zero. To calculate the sectional curvature, for each 2-dimensional tangent plane, corresponding to \((i \neq j)\), we must compute \( R_{ijij} = g_{js} R^s_{ijij} \). But \( g_{js} = 0 \) for \( s \neq j \), so \( R_{ijij} = g_{jj} R^j_{ijij} \). So,

\[
R^j_{ijij} = \frac{\partial}{\partial x^j} \Gamma^j_{ii} - \frac{\partial}{\partial x^i} \Gamma^j_{ij} + \Gamma^p_{ii} \Gamma^j_{pj} - \Gamma^p_{ij} \Gamma^j_{pi}.
\]

Then we can quickly compute that

\[
R_{121} = \Gamma^3_{11} \Gamma^2_{32} = -\frac{1}{x_3^2}.
\]

Similarly,

\[
R_{323} = R^1_{313} = -\frac{1}{x_3^2}.
\]

Thus, the sectional curvature \( K_{ij} \) is

\[
K_{ij} = \frac{g_{ii} R_{ijij}}{\det(g_{ii} g_{jj})} = -\frac{1}{x_3^2} = -1.
\]

Then by Equation 1.10, we have

\[
R_{ijkl} = -1 (g_{ik} g_{jl} - g_{il} g_{jk}). \quad (2.1)
\]
2.5 Geometry of Horospheres

Here, we discuss the geometry of horospheres in $\mathbb{H}^3$. In a sense, horospheres in $\mathbb{H}^3$ are analogous to planes in $\mathbb{R}^3$. This connection is made more clear in Chapter 3 with Volkov and Vladimirova's result that says all isometricall homeomorphisms of the Euclidean plane into hyperbolic 3-space are horospheres. Figure 2.3 shows a horosphere as a sphere in the Poincaré ball model with one point at infinity and as a plane in the upper half space model.

Figure 2.3: A horosphere in the Poincaré ball model and its image under the spherical inversion mapping to the upper half space model.

We will consider the upper half space model to make computations easier. As such, a horosphere is a plane at a fixed $x_3 = z_0$:

$$H = \{ x \in \mathbb{R}^3_+ \mid x_3 = z_0 \}.$$  

The metric on $H$ is given by

$$h = g \left| \frac{dx_i}{x_3^2} \right|^2 = \frac{dx_1^2 + dx_2^2}{z_0^2}.$$  

We will first calculate the principal curvatures of $H$ by finding the eigenvalues of the shape operator $\nabla \nabla = h^{ik} II_{ij}$. We adopt the convention of taking the upward normal vector to $H$ in the upper half space model, that is

$$\mathbf{n} = (0, 0, z_0).$$  

The scaling factor of $z_0$ is required to make $\mathbf{n}$ normal with respect to the metric. Let $\nabla$ be the Levi-Civita connection on $\mathbb{H}^3$. Then by definition of the second fundamental form,

$$II_{ij} = g \left( \mathbf{n}, \nabla \frac{\partial}{\partial x^i} \nabla \frac{\partial}{\partial x^j} \right).$$  

In Section 2.4, we calculated the Christoffel symbols for the upper half space model $\mathbb{H}^3$ and found that the only nonzero Christoffel symbols were

$$\tilde{\Gamma}^1_{13} = \tilde{\Gamma}^2_{23} = \tilde{\Gamma}^3_{33} = -\frac{1}{x_3},$$

$$\tilde{\Gamma}^3_{11} = \tilde{\Gamma}^3_{22} = \frac{1}{x_3}.$$  

Thus,

$$II_{ij} = g \left( \mathbf{n}, \tilde{\Gamma}^\alpha_{ij} \frac{\partial}{\partial x^\alpha} \right).$$
Noting that $x_3 = z_0$ on $H$, we can calculate

$$II_{12} = g((0,0,z_0),(0,0,0)) = 0$$
$$II_{11} = g\left((0,0,z_0),\left(0,0,\frac{1}{z_0}\right)\right) = \frac{1}{z_0^2}$$
$$II_{22} = g\left((0,0,z_0),\left(0,0,\frac{1}{z_0}\right)\right) = \frac{1}{z_0^2}.$$

Now the shape operator is given by $S^k_j = h^k_iII_{ij}$. But from above, we know $h^{ik} = z_0^2\delta^k_i$ and $II_{ij} = \frac{1}{z_0^2}\delta^i_j$. Thus, $S^k_j = z_0^2\delta^k_j\frac{1}{z_0^2}\delta^i_i = \delta^k_j$. That is,

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the principal curvatures of $H$ are the eigenvalues of $S$, so $k_1 = k_2 = 1$. Recall from linear algebra that eigenvectors of symmetric operators in an inner product space are orthogonal. Therefore, if we take our coordinates $(x^1, x^2)$ to be the orthonormal principal directions, we have

$$I = II = (dx^1)^2 + (dx^2)^2$$

for the horosphere.

### 2.6 Geometry of Equidistant Cylinders

We will now discuss the geometry of equidistant cylinders in $\mathbb{H}^3$. Like the horospheres to planes, the equidistant cylinders in $\mathbb{H}^3$ are analogous to cylinders in $\mathbb{R}^3$. In $\mathbb{R}^3$, cylinders are defined to be the surfaces equidistant to lines. Noticing that lines in $\mathbb{R}^3$ are the geodesics, it is natural to extend the idea of equidistant cylinders to non-Euclidean space by considering the surfaces generated by equidistance from a geodesic. So let $\gamma$ be a geodesic in $\mathbb{H}^3$. Then the equidistant cylinder about $\gamma$ is given by

$$N = \{ v \in \mathbb{H}^3 \mid d(v, \gamma) = r \},$$

where $d$ is the hyperbolic distance in $\mathbb{H}^3$ and $r > 0$ is constant. Figure 2.4 shows an equidistant cylinder in the Poincaré ball model and the upper half space model.

![Figure 2.4: An equidistant cylinder in the Poincaré ball model and its image under the spherical inversion mapping to the upper half space model.](image)
We will use the cone in the upper half space model to compute the second fundamental form of the equidistant cylinder. Without loss of generality, assume that the origin of the cone is at $(0,0,0)$ and is the surface of revolution obtained by rotating the line with angle $\theta$ from the $z$-axis about the $z$-axis. This is shown in Figure 2.5.

Let $a = \tan \theta$. Then the coordinate patch of the equidistant cylinder in upper half space is given by

$$x(u,v) = (av \cos u, av \sin u, v); \quad u \in [0,2\pi), \ v \in (0, \infty).$$

Let $\frac{\partial}{\partial x^i}$ be the standard frame for the upper half space model as seen in Section 2.4. We wish to write an orthonormal frame for the equidistant cylinder in terms of this standard frame so that we can compute the second fundamental form. To wit,

$$x_1 = (-av \sin u, av \cos u, 0)$$
$$x_2 = (a \cos u, a \sin u, 1).$$

Recall that the metric in the upper half space is $g(u,v) = \frac{1}{4}\langle u,v \rangle$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $u, v$ are tangent vectors at a point $(x_1, x_2, x_3) \in \mathbb{R}^3_+$. In our parametrization for the equidistant cylinder, $x_3 = v$. Thus, normalizing $x_1$ and $x_2$, we take

$$\frac{\partial}{\partial y^1} = \frac{x_1}{||x_1||} = (-v \sin u, v \cos u, 0)$$
$$\frac{\partial}{\partial y^2} = \frac{x_2}{||x_2||} = \frac{1}{\sqrt{1+a^2}} (av \cos u, av \sin u, v)$$

to be the basis for $T_pN$. The unit normal $n$ is given by

$$n = \frac{\partial}{\partial y^1} \times \frac{\partial}{\partial y^2} = \frac{1}{\sqrt{1+a^2}} (v \cos u, v \sin u, -av).$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\mathbb{R}^3$. Then

$$II_{ij} = g \left( n, \bar{\nabla} \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \right).$$

Let us compute $II_{11}$ first. We can write $\frac{\partial}{\partial y^i}$ in terms of the standard frame as

$$\frac{\partial}{\partial y^1} = -v \sin u \frac{\partial}{\partial x^1} + v \cos u \frac{\partial}{\partial x^2}. $$
Then recalling the Christoffel symbols previously calculated in Section 2.4, we have
\[
\nabla \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_1} = -v \sin u \nabla \frac{\partial}{\partial x_1} + v \cos u \nabla \frac{\partial}{\partial x_2}
\]
\[
= -v \sin u \left( -v \sin u \nabla \frac{\partial}{\partial x_1} + v \cos u \nabla \frac{\partial}{\partial x_2} \right)
\]
\[
+ v \cos u \left( -v \sin u \nabla \frac{\partial}{\partial x_2} + v \cos u \nabla \frac{\partial}{\partial x_2} \right)
\]
\[
= v \frac{\partial}{\partial x^3}.
\]
Thus,
\[
II_{11} = g \left( n, \nabla \frac{\partial}{\partial y_1} \right) = -\frac{a}{\sqrt{1 + a^2}} = -\sin \theta.
\]
Similar calculations give
\[
II_{12} = 0 \quad \text{and} \quad II_{22} = -\frac{1}{\sin \theta}.
\]
Thus,
\[
II = \begin{pmatrix}
-\sin \theta & 0 \\
0 & -\frac{1}{\sin \theta}
\end{pmatrix}
\]

The shape operator \( S^k \) is given by \( S^k_j = h^{ik}II_{ij} \), where \( h = g|_N \). Now \( h_{ij} = g \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right) \), so
\[
h = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
Thus,
\[
S = \begin{pmatrix}
-\sin \theta & 0 \\
0 & -\frac{1}{\sin \theta}
\end{pmatrix}
\]
Therefore, the principle curvatures are \( k_1 = b \) and \( k_2 = \frac{1}{b} \), where \( b = - \sin \theta \). Taking our coordinates \((x^1, x^2)\) to be the orthonormal vectors in the principle directions (along the meridians and parallels) give
\[
I = (dx^1)^2 + (dx^2)^2 \quad \text{and} \quad II = b(dx^1)^2 + \frac{1}{b}(dx^2)^2.
\]
Chapter 3

Isometric Immersion of $\mathbb{R}^2$ into $\mathbb{H}^3$

3.1 Volkov and Vladimirova’s Result

We can finally state and prove Volkov and Vladimirova’s result on the isometric immersion of $\mathbb{R}^2$ into $\mathbb{H}^3$.

Theorem 8 (Volkov and Vladimirova [8]). Let $\phi : \mathbb{R}^2 \to \mathbb{H}^3$ be an isometric immersion of the Euclidean plan $\mathbb{R}^2$ into hyperbolic 3-space $\mathbb{H}^3$. Then one of the following two situations is possible:

1. $\phi$ is a homeomorphism, and $\phi(\mathbb{R}^2)$ is a horosphere;
2. $\phi$ is a locally isometric covering of the surface formed by the rotation of a particular equidistant about its base line (that is, an equidistant cylinder).

Proof. For notational sake, let $M = \phi(\mathbb{R}^2)$ and denote the pullback metric on $M$ as $g$. Let $(x^1, x^2)$ be local coordinates on $M$. Since $\phi$ is an isometry of the flat plane, $I = g = (dx^1)^2 + (dx^2)^2$. Thus, by the fundamental theorem of surfaces, it suffices to find the second fundamental form $II$ for $M$ to characterize the immersed surface.

Let $L_{ij}, \Gamma_{ij}^k, R_{ijkl}$ for $i, j, k, l = 1, 2$ be the coefficients of the second fundamental form, the Christoffel symbols, and the coefficients of the Riemann curvature tensor on $M$. Let $\tilde{R}_{ijk}$ for $i, j, k, l = 1, 2, 3$ be the coefficients of the Riemann curvature tensor of $\mathbb{H}^3$. Then the Gauss and Codazzi-Mainardi equations (Theorem 6) state

$$\frac{\partial}{\partial x^1} L_{12} - \frac{\partial}{\partial x^2} L_{11} - \Gamma_{11}^s L_{s2} + \Gamma_{12}^s L_{s1} = -\tilde{R}_{1213}$$
$$\frac{\partial}{\partial x^1} L_{22} - \frac{\partial}{\partial x^2} L_{21} - \Gamma_{21}^s L_{s2} + \Gamma_{22}^s L_{s1} = -\tilde{R}_{2213}$$

$$L_{11} L_{22} - L_{12}^2 = R_{1212} - \tilde{R}_{1212}$$

But, since $\mathbb{H}^3$ has constant curvature $-1$, we have

$$\tilde{R}_{ijkl} = -1 (\tilde{g}_{ik} \tilde{g}_{jl} - \tilde{g}_{il} \tilde{g}_{jk}).$$

Assuming orthonormality of the coordinate system on $\mathbb{H}^3$ with respect to the metric $\tilde{g}$, we see that

$$\tilde{R}_{1213} = \tilde{R}_{2213} = 0;$$
$$\tilde{R}_{1212} = -1(\tilde{g}_{11} \tilde{g}_{22} - \tilde{g}_{12}^2) = -1.$$
Moreover, since $\phi$ is an isometry and $\mathbb{R}^2$ is flat, $\Gamma^k_{ij} = R_{ijkl} = 0$. Thus, we can rewrite the Gauss and Codazzi-Mainardi equations as

\[
\frac{\partial}{\partial x^1} L_{12} - \frac{\partial}{\partial x^2} L_{11} = 0 \tag{3.1}
\]
\[
\frac{\partial}{\partial x^1} L_{22} - \frac{\partial}{\partial x^2} L_{21} = 0 \tag{3.2}
\]
\[
L_{11} L_{22} - L_{12}^2 = 1 \tag{3.3}
\]

From Equation 3.1, we have

\[
L_{12} = \int \frac{\partial}{\partial x^2} L_{11} \, dx^1
\]
\[
L_{11} = \int \frac{\partial}{\partial x^1} L_{12} \, dx^2.
\]

Integrating the above equations gives a function $\varphi(x^1, x^2)$ defined by

\[
\varphi = \frac{1}{2} \left[ \int L_{12} \, dx^2 + \int L_{11} \, dx^1 \right],
\]

with $\frac{\partial}{\partial x^1} \varphi = L_{11}$ and $\frac{\partial}{\partial x^2} \varphi = L_{12}$. For sake of notation, we will denote the partials as subscripts, so

$\varphi_1 = L_{11}$ and $\varphi_2 = L_{12}$.

By similar construction from Equation 3.2, there exists $\psi(x^1, x^2)$ such that

$\psi_1 = L_{12}$ and $\psi_2 = L_{22}$.

Notice that $\varphi_2 = \psi_1$ and $\varphi, \psi$ are defined on all of $\mathbb{R}^2$.

From vector calculus, we know that if a vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in a simply connected domain $D \in \mathbb{R}^2$, and

\[
\frac{\partial F_2}{\partial x^1} - \frac{\partial F_1}{\partial x^2} = 0,
\]

then $F$ is a conservative vector field in $D$.

Define $F(x^1, x^2) : \mathbb{R}^2 \to \mathbb{R}^2$ by

\[
F(x^1, x^2) = (\varphi(x^1, x^2), \psi(x^1, x^2)).
\]

Since $\varphi$ and $\psi$ are continuously differentiable, $F$ is continuously differentiable. Moreover,

\[
\frac{\partial F_2}{\partial x^1} - \frac{\partial F_1}{\partial x^2} = \psi_1 - \varphi_2 = 0
\]

by construction of $\varphi$ and $\psi$. Thus, $F$ is indeed a conservative vector field on $\mathbb{R}^2$. Thus, there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = F$. In other words,

$\varphi_1 = f_1$ and $\psi_2 = f_2$.

Moreover,

$\varphi_2 = L_{12}$; $\psi_2 = L_{22}$.

Equation 3.3 then states that

$\varphi_{11} \varphi_{22} - (\varphi_{12})^2 = 1$ on $\mathbb{R}^2$. \tag{3.4}
By Jörgen’s theorem [2], any solution to Equation 3.4 on the entirety of $\mathbb{R}^2$ is a second degree polynomial. Thus, we can conclude that $f_{ij} = L_{ij}$ is constant. Without loss of generality, we can rotate the coordinates $(x^1, x^2)$ so that $L_{12} = 0$. Thus, the solution to Equation 3.3 has the form

$$L_{11} = \frac{1}{L_{22}} = a; \quad L_{12} = 0$$

where $a$ is a constant.

In the case where $a = 1$, we have $L_{11} = L_{22} = 1$ and $L_{12} = 0$, so the first and second fundamental forms of $M$ are

$$I = II = (dx^1)^2 + (dx^2)^2.$$

Thus, $M$ is a horosphere by the Bonnet theorem and the results in Section 2.5.

Otherwise, for $a \neq 1$, the first and second fundamental forms of $M$ are

$$I = (dx^1)^2 + (dx^2)^2 \quad \text{and} \quad II = a(dx^1)^2 + \frac{1}{a}(dx^2)^2.$$

Thus, by the Bonnet theorem and the results in Section 2.6, $M$ is an equidistant cylinder.

Note that in both cases, we are guaranteed a global isometry between the immersed plane and either the horosphere or the equidistant cylinder since $\mathbb{R}^2$ is simply connected. $\square$
Bibliography


