Section 2.8

[1] Given equation \( y = x^2 - \sqrt{x} \). Differentiating both sides with respect to \( t \) we have

\[
\frac{dy}{dt} = \frac{d}{dt}[x^2 - \sqrt{x}]
= 2x \frac{dx}{dt} - \frac{1}{2} x^{-1/2} \frac{dx}{dt}
= \left[ 2x - \frac{1}{2\sqrt{x}} \right] \frac{dx}{dt}.
\]

(1)

(a) Plugging in \( x = 4 \) and \( \frac{dx}{dt} = 8 \) in the above equation (1) we get

\[
\frac{dy}{dt} = \left[ 8 - \frac{1}{2\sqrt{4}} \right] \times 8
= \left[ 8 - \frac{1}{4} \right] \times 8
= 64 - 2 = 62.
\]

(b) Plugging in \( x = 16 \) and \( \frac{dy}{dt} = 12 \) in (1) we obtain

\[
12 = \left[ 32 - \frac{1}{2\sqrt{16}} \right] \frac{dx}{dt}
\]

i.e., \( 12 = \left[ 32 - \frac{1}{8} \right] \frac{dx}{dt} \)

i.e., \( 12 = \frac{255}{8} \frac{dx}{dt} \)

i.e., \( \frac{dx}{dt} = \frac{96}{255} = \frac{32}{85} \).

[4] \( x^2 + y^2 = 25 \). Differentiating both sides with respect to \( t \) we get

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]

i.e., \( x \frac{dx}{dt} + y \frac{dy}{dt} = 0 \).
(a) Putting $x = 3, y = 4,$ and $\frac{dx}{dt} = 8$ in (2) we get
\[24 + 4\frac{dy}{dt} = 0\]
i.e., \[4\frac{dy}{dt} = -24\]
i.e., \[\frac{dy}{dt} = -6.\]

(b) Putting $x = 4, y = 3,$ and $\frac{dy}{dt} = -2$ in (2) we get
\[4\frac{dx}{dt} - 6 = 0\]
i.e., \[4\frac{dx}{dt} = 6\]
i.e., \[\frac{dx}{dt} = \frac{6}{4} = \frac{3}{2}.\]

[10] Volume of the cone is $V = \frac{1}{3}\pi r^2 h$. But we know that $h = 3r$. Therefore $V = \frac{1}{3}\pi r^2 \times 3r = \pi r^3$. Now differentiating both sides with respect to $t$ we have
\[\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}.\quad (3)\]

We know that the radius of the cone is increasing at a rate of 2 inches per minute. Mathematically it means that $\frac{dr}{dt} = 2$ in/min.

(a) To find the rate of change of the volume when $r = 6$ inches, we put $r = 6$ and $\frac{dr}{dt} = 2$ in the equation (3). Then we obtain
\[\frac{dV}{dt} = 3\pi \times 36 \times 2 = 216\pi\]
Therefore the rate of change of volume at $r = 6$ inches is $216\pi$ cubic in/min.

(b) Similarly, when $r = 24$ inches we have
\[\frac{dV}{dt} = 3\pi \times (24)^2 \times 2 = 3456\pi.\]
Therefore the rate of change of volume at $r = 24$ inches is $3456\pi$ cubic in/min.

[14] Suppose the length of each edge of the cube is $x$ cm. [Since the cube is expanding, the edge length $x$ is also increasing. Therefore $x$ depends on ‘time’ ($t$) i.e., $x$ is a function of time ($t$).] Notice that there are total six surface planes of a cube, and area of each surface plane is $x^2$ (because edge length is $x$). Therefore total surface area of the cube is given by $S = 6x^2$. 
We want to find the rate of change of surface area with respect to time, i.e., we have to find \( \frac{dS}{dt} \). Differentiating both side of the equation \( S = 6x^2 \) with respect to \( t \) (\( t \) stands for time), we have

\[
\frac{dS}{dt} = 12x \frac{dx}{dt}.
\]  

(4)

We know that edges are expanding at the rate of 3cm/sec. In other words, \( \frac{dx}{dt} = 3 \text{cm/sec} \).

(a) Now putting \( x = 1 \) and \( \frac{dx}{dt} = 3 \) in equation (4) we get

\[
\frac{dS}{dt} = 12 \times 3 = 36.
\]

Therefore the surface area of the cube is increasing at the rate of 36 square cm/sec when each edge is 1 cm.

(b) Similarly, putting \( x = 10 \) and \( \frac{dx}{dt} = 3 \) in equation (4) we get

\[
\frac{dS}{dt} = 120 \times 3 = 360.
\]

Therefore the surface area of the cube is increasing at the rate of 360 square cm/sec when each edge is 10 cm.

[16] Given equation \( y = \frac{1}{1+x^2} \). Rewrite this equation as \( y = (1 + x^2)^{-1} \). Differentiating both sides with respect to \( t \) we have

\[
\frac{dy}{dt} = \frac{d}{dt} \left[(1 + x^2)^{-1}\right]
= -(1 + x^2)^{-2} \frac{d}{dt}[1 + x^2]
= -(1 + x^2)^{-2} 2x \frac{dx}{dt}.
\]

(5)

We know that \( \frac{dx}{dt} = 2 \text{cm/min} \).

(a) Plugging in \( x = -2 \) and \( \frac{dx}{dt} = 2 \) in (5) we have

\[
\frac{dy}{dt} = -(1 + 4)^{-2} \times (-4) \times 2
= -\frac{1}{5^2} \times (-8)
= \frac{8}{25} \text{cm/min}.
\]

(b) Similarly, plugging in \( x = 2 \) and \( \frac{dx}{dt} = 2 \) in (5) we have

\[
\frac{dy}{dt} = -\frac{8}{25} \text{cm/min}.
\]
(c) When $x = 0$

\[
\frac{dy}{dt} = -(1 + 0)^{-2} \times 0 = 0 \text{ cm/min.}
\]

(d) And when $x = 10$

\[
\begin{align*}
\frac{dy}{dt} &= -(1 + 100)^{-2} \times 20 \times 2 \\
&= -\frac{40}{101^2} \text{ cm/min.}
\end{align*}
\]

[17] Let $r$ be the height of the ladder along the wall (see figure), and the base of the ladder be at a distance $x$ from the wall. By Pythagorean theorem we have $r^2 + x^2 = 25^2$.

![Diagram of ladder leaning against a wall with a rate of 2 feet per second pulling the base of the ladder away from the wall.](image)

We know that base of the ladder is pulled away from the house at a rate of 2 feet per second i.e., $\frac{dx}{dt} = 2 \text{ ft/sec}$. We want to find how fast the top of the ladder is moving down i.e, we have to find $\frac{dr}{dt}$. Now differentiating both sides of $r^2 + x^2 = 25^2$ with respect to $t$ we get

\[
2r \frac{dr}{dt} + 2x \frac{dx}{dt} = 0
\]

\[
\text{i.e., } r \frac{dr}{dt} + x \frac{dx}{dt} = 0 \quad (6)
\]

(a) When $x = 7$, we have $r = \sqrt{25^2 - 7^2} = 24$. Putting $x = 7$, $r = 24$, and $\frac{dx}{dt} = 2$ in the equation $(6)$ we obtain

\[
\frac{24}{dt} + 14 = 0
\]

\[
\text{i.e., } \frac{dr}{dt} = -\frac{14}{24} = -\frac{7}{12}.
\]
Therefore when the base is 7 feet away from the house, the top of the ladder is moving down at the rate of \( \frac{7}{12} \) ft/sec.

(b) Similarly, when \( x = 15 \) we have \( r = \sqrt{25^2 - 15^2} = 20 \). Putting \( x = 15, \ r = 20, \) and \( \frac{dx}{dt} = 2 \) in the equation (6) we obtain

\[
20 \frac{dr}{dt} + 30 = 0
\]

i.e.,

\[
\frac{dr}{dt} = -\frac{30}{20} = -\frac{3}{2}.
\]

Therefore when the base is 15 feet away from the house, the top of the ladder is moving down at the rate of \( \frac{3}{2} \) ft/sec.

(c) When \( x = 24 \), we have \( r = \sqrt{25^2 - 24^2} = 7 \). Putting \( x = 24, \ r = 7, \) and \( \frac{dx}{dt} = 2 \) in the equation (6) we obtain

\[
7 \frac{dr}{dt} + 48 = 0
\]

i.e.,

\[
\frac{dr}{dt} = -\frac{48}{7}.
\]

Therefore when the base is 24 feet away from the house, the top of the ladder is moving down at the rate of \( \frac{48}{7} \) ft/sec.

[19] Solved in class.