Section 3.1

[1] \( f(x) = \frac{x^2}{x^2+4} \). Differentiating with respect to \( x \) we have

\[
 f'(x) = \frac{d}{dx} \left[ \frac{x^2}{x^2+4} \right]
 = \frac{(x^2 + 4) \frac{d}{dx}[x^2] - x^2 \frac{d}{dx}[x^2 + 4]}{(x^2 + 4)^2}
 = \frac{2x(x^2 + 4) - x^2[2x + 0]}{(x^2 + 4)^2}
 = \frac{2x^3 + 8x - 2x^3}{(x^2 + 4)^2}
 = \frac{8x}{(x^2 + 4)^2}.
\]

Given points are \((0, 0)\), \((1, \frac{1}{5})\), and \((-1, \frac{1}{5})\). Values of \( f'(x) \) at those points are

\[
 f'(0) = \frac{0}{(0+4)^2} = \frac{0}{16} = 0,
\]

\[
 f'(1) = \frac{8}{(1+4)^2} = \frac{8}{5^2} = \frac{8}{25},
\]

and

\[
 f'(-1) = \frac{-8}{(1+4)^2} = \frac{-8}{5^2} = -\frac{8}{25}.
\]
[4] Given function \( f(x) = -3x\sqrt{x+1} \). Differentiating with respect to \( x \) we have

\[
\begin{align*}
    f'(x) &= \frac{d}{dx}[-3x\sqrt{x+1}] \\
         &= \frac{d}{dx}[-3x]\sqrt{x+1} - 3x \frac{d}{dx}\sqrt{x+1} \\
         &= -3\sqrt{x+1} - 3x \frac{d}{dx}(x+1)^{1/2} \\
         &= -3\sqrt{x+1} - 3x \times \frac{1}{2}(x+1)^{-1/2} \frac{d}{dx}[x+1] \\
         &= -3\sqrt{x+1} - \frac{3x}{2\sqrt{x+1}}.
\end{align*}
\]

Given points are \((-1, 0)\), \((-\frac{2}{3}, \frac{2\sqrt{3}}{3})\), and \((0, 0)\). Values of \( f'(x) \) at those points are

\[
\begin{align*}
    f'(-1) &= 3\sqrt{-1+1} - \frac{3(-1)}{2\sqrt{-1+1}} \\
          &= 3 \times 0 + \frac{3}{2 \times 0} \\
          &= \frac{3}{0}.
\end{align*}
\]

Since \( \frac{3}{0} \) is undefined, \( f'(0) \) is undefined.

\[
\begin{align*}
    f'\left(-\frac{2}{3}\right) &= -3\sqrt{-\frac{2}{3}+1} - \frac{3 \times \left(-\frac{2}{3}\right)}{2\sqrt{-\frac{2}{3}+1}} \\
                             &= -3\sqrt{-\frac{1}{3}} + \frac{2}{2\sqrt{-\frac{1}{3}}} \\
                             &= -3 \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\
                             &= -\sqrt{3} + \sqrt{3} \\
                             &= 0.
\end{align*}
\]

\[
\begin{align*}
    f'(0) &= -3\sqrt{0+1} - \frac{0}{2\sqrt{0+1}} \\
          &= -3 \cdot 0 + \frac{0}{2} \\
          &= -3.
\end{align*}
\]
[7] $f(x) = x^4 - 2x^2$. First of all, we have to find the critical points. Differentiating $f(x)$ with respect to $x$ we have

$$f'(x) = 4x^3 - 4x.$$ 

To find critical points we solve

$$f'(x) = 0$$

i.e., $4x^3 - 4x = 0$

i.e., $x^3 - x = 0$

i.e., $x(x^2 - 1) = 0$

i.e., $x = 0, x^2 - 1 = 0$

i.e., $x = 0, x^2 = 1$

i.e., $x = 0, x = ±1$.

Also notice that $f'(x)$ is defined everywhere. Therefore $x = 0, ±1$. We do the following to determine the open intervals on which the function is increasing or decreasing

<table>
<thead>
<tr>
<th>Test intervals</th>
<th>$−∞ &lt; x &lt; −1$</th>
<th>$−1 &lt; x &lt; 0$</th>
<th>$0 &lt; x &lt; 1$</th>
<th>$1 &lt; x &lt; ∞$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Text points</td>
<td>$x = −2$</td>
<td>$x = −\frac{1}{2}$</td>
<td>$x = \frac{1}{2}$</td>
<td>$x = 2$</td>
</tr>
<tr>
<td>Sign of $f'(x)$</td>
<td>$f'(-2) = −24 &lt; 0$</td>
<td>$f'(-\frac{1}{2}) = \frac{3}{2} &gt; 0$</td>
<td>$f'(\frac{1}{2}) = −\frac{3}{2} &lt; 0$</td>
<td>$f'(2) = 24 &gt; 0$</td>
</tr>
<tr>
<td>Conclusion</td>
<td>Decreasing</td>
<td>Increasing</td>
<td>Decreasing</td>
<td>Increasing</td>
</tr>
</tbody>
</table>

[8] Given function $f(x) = \frac{x^2}{x+1}$. Differentiating with respect to $x$ we have

$$f'(x) = \frac{d}{dx} \left[ \frac{x^2}{x+1} \right]$$

$$= \frac{(x+1) \frac{d}{dx}[x^2] - x^2 \frac{d}{dx}[x+1]}{(x+1)^2}$$

$$= \frac{2x(x+1) - x^2}{(x+1)^2}$$

$$= \frac{2x^2 + 2x - x^2}{(x+1)^2}$$

$$= \frac{x^2 + 2x}{(x+1)^2}.$$ 

To solve the equation $f'(x) = 0$ we have

$$\frac{x^2 + 2x}{(x+1)^2} = 0$$

i.e., $x^2 + 2x = 0$

i.e., $x(x + 2) = 0$

i.e., $x = 0, −2$. 

3
Notice that \( f'(x) \) is undefined for \( x = -1 \), because \( f'(-1) = \frac{1-2}{0} = \frac{-1}{0} \). But \( f(x) \) is also undefined for \( x = -1 \). Therefore \( x = -1 \) is not a critical number, and the only critical numbers are \( x = 0, -2 \).

Now we proceed as following.

\[
\begin{array}{|c|c|c|}
\hline
\text{Test intervals} & -\infty < x < -2 & -2 < x < 0 & 0 < x < \infty \\
\hline
\text{Test points} & x = -3 & x = -\frac{1}{2} & x = 1 \\
\hline
\text{Sign of } f'(x) & f'(-3) = \frac{3}{4} > 0 & f'(-\frac{1}{2}) = -3 < 0 & f'(1) = \frac{3}{4} > 0 \\
\hline
\text{Conclusion} & \text{Increasing} & \text{Decreasing} & \text{Increasing} \\
\end{array}
\]

[17] Given function \( f(x) = \sqrt{x^2 - 1} \). Differentiating with respect to \( x \) we obtain

\[
f'(x) = \frac{d}{dx}[(x^2 - 1)^{1/2}] = \frac{1}{2}(x^2 - 1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 - 1}}.
\]

Notice that there is no real solution of \( f'(x) = 0 \).

[Common mistake:

\[
x \sqrt{x^2 - 1} = 0
\]

\[i.e., \quad x = 0.
\]

But if we plugin \( x = 0 \) in \( f'(x) \) we get \( f'(0) = \frac{0}{\sqrt{-1}} \), which is not a real number. Because we have \( \sqrt{\text{negative number}} \).]

Now we observe that \( f'(x) \) is undefined if

\[
\sqrt{x^2 - 1} = 0
\]

\[i.e., \quad x^2 - 1 = 0
\]

\[i.e., \quad x^2 = 1
\]

\[i.e., \quad x = \pm 1.
\]

We also notice that \( f(\pm 1) = \sqrt{1-1} = 0 \) i.e., \( f(x) \) is well defined for \( x = \pm 1 \). Therefore \( x = \pm 1 \) are critical points. Ideally the test intervals should be \( -\infty < x < -1, -1 < x < 1, \) and \( 1 < x < \infty \). But the given function \( f(x) = \sqrt{x^2 - 1} \) is undefined for \( x^2 - 1 < 0 \) (because negative number inside a square root is not allowed) i.e., for \( -1 < x < 1 \). So it does not make sense to talk about increasing or decreasing inside the interval \(-1 < x < 1 \). Therefore we have the following

\[
\begin{array}{|c|c|}
\hline
\text{Test intervals} & -\infty < x < -1 & 1 < x < \infty \\
\hline
\text{Test points} & x = -2 & x = 2 \\
\hline
\text{Sign of } f'(x) & f'(-2) = -\frac{2}{\sqrt{3}} < 0 & f'(2) = \frac{2}{\sqrt{3}} > 0 \\
\hline
\text{Conclusion} & \text{Decreasing} & \text{Increasing} \\
\end{array}
\]
[18] the given function \( f(x) = \sqrt{4-x^2} \). Differentiating with respect to \( x \) we get

\[
f'(x) = \frac{d}{dx}[(4 - x^2)^{1/2}] \\
= \frac{1}{2} (4 - x^2)^{-1/2} \frac{d}{dx}[4 - x^2] \\
= \frac{1}{2} (4 - x^2)^{-1/2} (-2x) \\
= -\frac{x}{\sqrt{4-x^2}}.
\]

to solve the equation \( f'(x) = 0 \) we have

\[
-\frac{x}{\sqrt{4-x^2}} = 0 \\
i.e., \quad -x = 0 \\
i.e., \quad x = 0.
\]

[Note that this problem is slightly different from the previous one. Plugging in \( x = 0 \) in \( f'(x) \) we have \( f'(0) = -\frac{0}{\sqrt{4}} = -\frac{0}{2} = 0 \). Therefore \( x = 0 \) is indeed a solution of \( f'(x) = 0 \).]

We also notice that \( f'(x) \) is undefined for

\[
\sqrt{4-x^2} = 0 \\
i.e., \quad 4-x^2 = 0 \\
i.e., \quad 4 = x^2 \\
i.e., \quad x = \pm 2.
\]

Whereas \( f(\pm 2) = \sqrt{4-4} = 0 \) i.e., \( f(x) \) is well defined for \( x = \pm 2 \). Hence the critical points are \( x = 0, \pm 2 \). Ideally the test intervals should be \(-\infty < x < -2, -2 < x < 0, 0 < x < 2, \) and \( 2 < x < \infty \). But observe that the given function \( f(x) = \sqrt{4-x^2} \) is undefined for \( 4-x^2 < 0 \) (because negative number inside a square root is not allowed) i.e., \( 4 < x^2 \) i.e., \( 2 < x \) or \( x < -2 \). In other words the function is well defined if \( 4-x^2 \geq 0 \) i.e., if \( x^2 \leq 4 \) i.e., if \( -2 \leq x \leq 2 \). Therefore it makes sense to talk about increasing or decreasing only when \( -2 \leq x \leq 2 \). Consequently we have the following

<table>
<thead>
<tr>
<th>Test intervals</th>
<th>(-2 &lt; x &lt; 0)</th>
<th>(0 &lt; x &lt; 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test points</td>
<td>(x = -1)</td>
<td>(x = 1)</td>
</tr>
<tr>
<td>Sign of (f'(x))</td>
<td>(f'(-1) = \frac{1}{\sqrt{3}} &gt; 0)</td>
<td>(f'(1) = -\frac{1}{\sqrt{3}} &lt; 0)</td>
</tr>
<tr>
<td>Conclusion</td>
<td>Increasing</td>
<td>Decreasing</td>
</tr>
</tbody>
</table>
Given function \( f(x) = x\sqrt{x + 1} \). Differentiating with respect to \( x \) we get

\[
f'(x) = \frac{d}{dx}[x\sqrt{x + 1}] = \frac{d}{dx}[x]\sqrt{x + 1} + x \frac{d}{dx}[\sqrt{x + 1}]
\]

\[
= \sqrt{x + 1} + x \frac{1}{2}(x + 1)^{-\frac{1}{2}} \frac{d}{dx}[x + 1]
\]

\[
= \sqrt{x + 1} + \frac{x}{2\sqrt{x + 1}}
\]

To find the critical points we solve

\[
f'(x) = 0
\]

i.e., \( \sqrt{x + 1} + \frac{x}{2\sqrt{x + 1}} = 0 \)

i.e., \( \sqrt{x + 1} = -\frac{x}{2\sqrt{x + 1}} \)

i.e., \( \sqrt{x + 1} \cdot 2\sqrt{x + 1} = -x \)

i.e., \( 2(x + 1) = -x \)

i.e., \( 2x + 2 = -x \)

i.e., \( 3x + 2 = 0 \)

i.e., \( x = -\frac{2}{3} \).

We also notice that \( f'(x) \) is undefined for \( x = -1 \) (because \( f'(-1) = \sqrt{-1 + 1} + \frac{-1}{2\sqrt{-1 + 1}} = \frac{-1}{0} \)). Whereas \( f(-1) = (-1) \cdot \sqrt{-1 + 1} = (-1) \times 0 = 0 \) i.e., \( f(x) \) is well defined for \( x = -1 \). Therefore \( x = -1 \) is also a critical point. Consequently, we have two critical points \( x = -\frac{2}{3}, -1 \).

Ideally the test intervals should be \(-\infty < x < -1\), \(-1 < x < -\frac{2}{3}\), and \(-\frac{2}{3} < x < \infty\). But notice that the given function \( f(x) = x\sqrt{x + 1} \) is undefined for \( x + 1 < 0 \) (because \( \sqrt{\text{negative number is not allowed}} \)) i.e., \( x < -1 \). Therefore we have the following

<table>
<thead>
<tr>
<th>Test intervals</th>
<th>(-1 &lt; x &lt; -\frac{2}{3})</th>
<th>(-\frac{2}{3} &lt; x &lt; \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test points</td>
<td>( x = -\frac{2}{3} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>Sign of ( f'(x) )</td>
<td>( f'(-\frac{2}{6}) = -\frac{3}{2\sqrt{6}} &lt; 0 )</td>
<td>( f'(0) = 1 &gt; 0 )</td>
</tr>
<tr>
<td>Conclusion</td>
<td>Decreasing</td>
<td>Increasing</td>
</tr>
</tbody>
</table>

Alternative method

We have \( f'(x) = \sqrt{x + 1} + \frac{x}{2\sqrt{x + 1}} \). We know that the function is increasing if \( f'(x) > 0 \)
and decreasing if $f'(x) < 0$. Therefore the given function $f(x) = x\sqrt{x+1}$ is increasing if

$$f'(x) > 0$$

**i.e.,**

$$\sqrt{x+1} + \frac{x}{2\sqrt{x+1}} > 0$$

**i.e.,**

$$\sqrt{x+1} > -\frac{x}{2\sqrt{x+1}}$$

**i.e.,**

$$2(x+1) > -x$$

**i.e.,**

$$2x + 2 + x > 0$$

**i.e.,**

$$3x + 2 > 0$$

**i.e.,**

$$3x > -2$$

**i.e.,**

$$x > -\frac{2}{3}.$$  

Therefore the function is increasing when $x > -\frac{2}{3}$. In other words, the function is increasing in the interval $-\frac{2}{3} < x < \infty$. Similarly the function is decreasing if

$$f'(x) < 0$$

**i.e.,**

$$\sqrt{x+1} + \frac{x}{2\sqrt{x+1}} < 0$$

**i.e.,**

$$x < -\frac{2}{3}.$$  

Therefore the function is decreasing in the interval $-\infty < x < -\frac{2}{3}$.  

But we know that the function $f(x) = x\sqrt{x+1}$ is not defined for $x+1 < 0$ i.e., $x < -1$. Therefore it does not make sense to talk about increasing or decreasing when $x < -1$. Consequently the function is decreasing in the interval $-1 < x < -\frac{2}{3}$ (not in the interval $-\infty < x < -\frac{2}{3}$).  

Finally, the function is increasing in the interval $-\frac{2}{3} < x < \infty$ and decreasing in the interval $-1 < x < -\frac{2}{3}$.

**Remark:** This alternative method can be used for any problem about increasing or decreasing function. There are a few advantages in this alternative method. We don’t have to compute test interval, test point, and sign of $f''(x)$ in each test interval.
Given function \( f(x) = \frac{x^2}{x^2+4} \). We compute

\[
f'(x) = \frac{d}{dx} \left[ \frac{x^2}{x^2+4} \right] = \left( \frac{x^2 + 4 \frac{d}{dx}[x^2] - x^2 \frac{d}{dx}[x^2 + 4]}{(x^2 + 4)^2} \right) = \frac{2x(x^2 + 4) - x^2 \cdot 2x}{(x^2 + 4)^2} = \frac{2x^3 + 8x - 2x^3}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}
\]

Solving the equation \( f'(x) = 0 \) we obtain \( x = 0 \). Notice that since always \( x^2 + 4 \neq 0 \), \( f'(x) \) is defined everywhere. So, there is only one critical point, namely \( x = 0 \). Also \( f(x) \) is defined everywhere. Consequently, we have the following

<table>
<thead>
<tr>
<th>Test intervals</th>
<th>(-\infty &lt; x &lt; 0)</th>
<th>(0 &lt; x &lt; \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test points</td>
<td>(x = -1)</td>
<td>(x = 1)</td>
</tr>
<tr>
<td>Sign of (f'(x))</td>
<td>(f'(-1) = -\frac{8}{25} &lt; 0)</td>
<td>(f'(1) = \frac{8}{25} &gt; 0)</td>
</tr>
<tr>
<td>Conclusion</td>
<td>Decreasing</td>
<td>Increasing</td>
</tr>
</tbody>
</table>

Given function

\[
f(x) = \begin{cases} 
3x + 1, & x \leq 1 \\
5 - x^2, & x > 1. 
\end{cases}
\]

Differentiating with respect to \( x \) we have

\[
f'(x) = \begin{cases} 
3, & x < 1 \quad \text{(notice, I have written \( x < 1 \) not \( x \leq 1 \))} \\
-2x, & x > 1. 
\end{cases}
\]

Solving \( f'(x) = 0 \) we obtain \( x = 0 \) when \( x > 1 \). But when \( x \) can not be 0 when \( x > 1 \). Therefore there is no solution of \( f'(x) = 0 \). On the other hand \( f'(x) \) is undefined for \( x = 1 \) but \( f(x) \) is well defined for \( x = 1 \). Therefore \( x = 1 \) is the only critical point.

<table>
<thead>
<tr>
<th>Test intervals</th>
<th>(-\infty &lt; x &lt; 1)</th>
<th>(1 &lt; x &lt; \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test points</td>
<td>(x = 0)</td>
<td>(x = 2)</td>
</tr>
<tr>
<td>Sign of (f'(x))</td>
<td>(f'(0) = 3 &gt; 0)</td>
<td>(f'(2) = -4 &lt; 0)</td>
</tr>
<tr>
<td>Conclusion</td>
<td>Increasing</td>
<td>Decreasing</td>
</tr>
</tbody>
</table>

**Remark:** If a function is defined piecewise, then take the joining points as critical points (like \( x = 1 \) in this problem)

Alternative method
We have

\[ f'(x) = \begin{cases} 
  3, & x < 1 \\
  -2x, & x > 1.
\end{cases} \]

\[ = \begin{cases} 
  \text{positive}, & x < 1 \\
  \text{negative}, & x > 1.
\end{cases} \]

Therefore the function is increasing in the interval \(-\infty < x < 1\) and decreasing in the interval \(1 < x < \infty\).