Proposal for Qualifying Exam

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Date: Friday, September 5, 2014
Time: 9:00 AM
Location: 2240 MSB

Proposed Research Talk

Title: Minimal-genus Heegaard splittings of complements of high-distance knots.

Abstract: Suppose $K$ is a knot in $S^3$ with bridge number $n$ and bridge distance greater than $2n$. We show that there are at most $\binom{2n}{n}$ minimal genus Heegaard splittings of $S^3 \setminus \eta(K)$. These splittings can be divided into two families. Two splittings from the same family become equivalent after one stabilization. If $K$ has bridge distance at least $4n$, then two splittings from different families become equivalent only after at least $n-1$ stabilizations.

Technical background: Let $M$ be a closed, orientable, connected 3-manifold. A Heegaard surface for $M$ is a closed, orientable, connected surface $\Sigma$ embedded in $M$ so that $M \setminus \Sigma$ is a pair of handlebodies $H_1, H_2$. The triple $(\Sigma, H_1, H_2)$ is a Heegaard splitting of $M$. The minimal genus of all such surfaces $\Sigma$ is the Heegaard genus of $M$. Two Heegaard splittings are isotopic or equivalent if there is an ambient isotopy of $M$ taking one of the surfaces to the other. Otherwise, they are distinct.

Let $\Gamma \subset H$ be a graph onto which the handlebody $H$ deformation retracts. Then $\Gamma$ is a spine of $H$. Removing a regular neighborhood of some (possibly empty, possibly disconnected) subgraph of $\Gamma$ contained in $H$ results in a compression body if every component of the subgraph has non-positive Euler characteristic. However, we assume that no component of the boundary of this regular neighborhood is spherical. The spine of this compression body is the remainder of $\Gamma$, union the boundary of the regular neighborhood which was removed. If $M$ is a compact orientable manifold with boundary, then $\partial M$ is a closed orientable surface, and thus a Heegaard splitting of $M$ can be defined as a triple $(\Sigma, W_1, W_2)$ where $W_i$ are compression bodies.

Let $(\Sigma, W_1, W_2)$ be a genus-$g$ Heegaard splitting of $M$. If $\gamma$ is a properly embedded arc in, say, $W_2$, which is parallel into $\Sigma$, let $W'_1 = W_1 \cup N(\gamma)$, and $W'_2 = W_2 \setminus N(\gamma)$, with $\Sigma' = \partial W'_2 = \partial W'_1$. Then $W'_2$ and $W'_1$ are still compression bodies, and $(\Sigma', W'_1, W'_2)$ is a genus $g+1$ Heegaard splitting for $M$ which is a stabilization of $(\Sigma, W_1, W_2)$. If $\Sigma$ was a $(g, n)$ bridge surface for a knot $K$, and $\gamma$ a bridge arc, then $\Sigma'$ is a meridional stabilization of $\Sigma$ and is a $(g+1, n-1)$-bridge surface. In 1933, Reidemeister [9] and Singer [12] showed independently that any two inequivalent Heegaard surfaces for $M$ have stabilizations which are equivalent. The minimal genus of this common stabilization is called the stable genus of the two surfaces.

It was not known until much later whether there is any upper bound to the stable genus in terms of the genera of the two Heegaard surfaces. In 1996, Rubinstein and Scharlemann [11] proved that, for a non-Haken 3-manifold $M$ containing two Heegaard surfaces of genus $p$ and $q$ respectively, with $p \geq q$, their stable genus $s(p,q)$ was at most $5p + 8q - 9$. However, it was not known until 2007 whether there were any examples even with $s(p,q) \geq 1$. Joel Hass, Abigail Thompson and William

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Thurston [6] showed that there exist manifolds with a Heegaard surface which must be stabilized to twice its original genus in order to switch the positions of the two handlebodies. This is an example of high stable genus for a notion of equivalence which requires the orientations of the surfaces to coincide as well. Their proof was geometric, and Jesse Johnson proved a combinatorial version in [8]. These same combinatorial methods can be adapted to construct examples of Heegaard surfaces which have a high stable genus under unoriented equivalence as well.

If $K \subset M$ is a knot, a bridge surface for $K$ is a Heegaard surface $\Sigma$ for $M$ such that $K$ intersects $\Sigma$ transversely and $K \cap H^\pm$ is a collection of simultaneously-boundary arcs in each handlebody, called bridge arcs, or $K \cap \Sigma = \emptyset$ and $K$ is parallel into $\Sigma$. If the genus of $\Sigma$ is $g$ and the number of points of $\Sigma \cap K$ is $2n$, then $\Sigma$ is a $(g,n)$ bridge surface. If $n$ is minimal over all surfaces of some fixed genus $g$, then $K$ is said to be a $(g,n)$-bridge knot. If $g = 0$, this is reduced to simply saying that $K$ is $n$-bridge, and $\Sigma$ is a bridge sphere.

If a simple closed curve $\gamma$ on a Heegaard or bridge surface (or, in fact, any separating, connected, orientable, compact, bicompressible surface) $\Sigma$ does not bound a disk in $\Sigma$ or a disk punctured once by $K \cap \Sigma$ in $\Sigma$, we say that $\gamma$ is essential. We define the curve complex $C(\Sigma)$, first introduced by Hempel in [5], of the surface $\Sigma$ as follows: let there be a vertex for each isotopy class of essential simple closed curve on $\Sigma$, and let any two vertices corresponding to curves which have disjoint representatives on $\Sigma$ be connected by an edge. Let $K_1$ and $K_2$ be the collections of vertices corresponding to curves that bound disks to either side of $\Sigma$, which are disjoint from $K$ in the case of a bridge splitting. Then the minimal length edge path in $C(\Sigma)$ between a vertex of $K_1$ and a vertex of $K_2$ is called the distance of $\Sigma$, and is denoted $d(\Sigma)$. If $\Sigma$ is a bridge sphere for a knot $K \subset S^3$ which realizes the minimal distance of all bridge spheres for $K$, then we say that the knot has distance $d(\Sigma)$, which is denoted as $d(K)$.

Let $M_K = S^3 \setminus \eta(K)$. In this talk, we outline a proof of the following statement, using the methods of [8], which explicitly constructs Heegaard splittings with a very high stable genus:

**Theorem 1.** Let $K \subset S^3$ be an $n$-bridge knot with bridge distance greater than $2n$. Then the Heegaard genus of $M_K$ is $n$, and there are at most $\binom{2n}{n}$ distinct minimal genus Heegaard splittings of $M_K$. These are divided into two families: two Heegaard splittings from the same family will have a stable genus of $n + 1$; if the bridge distance of $K$ is at least $4n$, two Heegaard splittings from differing families will have a stable genus $2n - 1$.

**Outline of proof:** It follows from [13, Theorem 10.3] that, given $K$ an $n$-bridge knot with bridge distance greater than $2n$, that the bridge sphere which realizes the bridge number is unique up to ambient isotopy. Given such a bridge sphere $S$, we can construct a Heegaard splitting for $M_K$ by meridionally stabilizing $S$ $n$ times, and considering the resulting genus-$n$ surface to be embedded in $M_K$. It also follows from the same result of Tomova that any genus-$n$ Heegaard surface in $M_K$ is a meridional stabilization of $S_K$, the restriction of $S$ to $M_K$.

If $K$ is given an orientation, and the $2n$ points of $S \cap K$ are labeled $1, \ldots, 2n$ by following the knot in the direction of this orientation, for any genus-$n$ Heegaard surface of $M_K$ each tube attached to $S_K$ can be said to have a left end and a right end. We show that the order in which the tubes were attached to $S_K$ is fixed by identifying the indices of the left ends of the tube. This shows that there are at most $\binom{2n}{n}$ different tubings, though some may be equivalent.

If $S_K$ is given an orientation, then any genus-$n$ Heegaard surface $\Sigma$ for $M_K$ gets an induced orientation as well, so that $\Sigma$ splits $M_K$ into two sides. Fixing such an orientation on $S_K$, whether $K$ is on one side or the other of the induced orientation on $\Sigma$ divides the set of Heegaard splittings into the two families of Theorem 1.

If $K$ lies on the same side of two splittings $\Sigma, \Sigma'$, we show that there exists a sequence of Heegaard splittings $\Sigma = \Sigma_1, \ldots, \Sigma_k = \Sigma'$, where $\Sigma_i$ and $\Sigma_{i+1}$ are equivalent after just one stabilization. Since stabilizations are unique, this implies that all of the surfaces with $K$ on a fixed side become equivalent after just one stabilization. The stabilization-destabilization step which forms this sequence is constructed by compressing along an annulus $A$ which has one boundary component which is a meridian of one tube of $\Sigma_i$, and the other boundary component on another tube of $\Sigma_i$. This is equivalent to first stabilizing along an arc $\gamma$ that runs from one tube to the other, and then destabilizing.
by compressing along the disk $\mathcal{A} \setminus \eta(\gamma)$. The fact that the tubes of $\Sigma_i$ are formed by meridional stabilization along bridge arcs ensures that this really is a destabilization.

If $K$ lies on opposite sides of two splittings $\Sigma, \Sigma'$, then we know that a common stabilization $\Sigma''$ of $\Sigma$ and $\Sigma'$ must be flippable, that is, there must be an ambient isotopy of $M_K$ which takes $\Sigma''$ to itself but with reversed orientation. We use the double sweep-out machinery of [8] to show that this criterion bounds the genus of $\Sigma''$ below by $\min\{2n - 1, \frac{1}{2}d(K)\}$.

It remains an open question for further research whether, and under what conditions, the $\binom{2n}{n}$ surfaces can be shown to all be distinct.

**Proposed Exam Syllabus**

- **Analysis** [7, §§1-6, 8, 12]
  - Metric and normed spaces
  - Space of continuous functions
  - Contraction mapping
  - Banach spaces
  - Hilbert Spaces
  - Bounded Linear Operators on a Hilbert Space
  - $L^p$ spaces
    * Hölder’s Inequality
    * Young’s Inequality
    * Fubini’s Theorem

- **Algebra** [1, §§1-5]
  - Groups
    * Lagrange’s Theorem, Isomorphism Theorems
    * Group actions on sets, the class formula
    * Sylow Theorems
    * Products, semidirect products
    * Classification of finite abelian groups.
  - Rings and modules
    * Ideals and quotient rings
    * Prime and maximal ideals
    * Modules over a ring
    * Products, coproducts in $R$-Mod
    * UFD’s, PID’s, Euclidean domains
    * Irreducibility in polynomial rings

- **Algebraic Topology** [4, §§1-3]
  - Fundamental Group and covering spaces
    * Seifert-van Kampen Theorem
    * Lifting properties, classification of covering spaces
  - Homology
    * Simplicial and singular homology groups
    * Exact sequences, excision
    * Computations, cellular homology, Mayer-Vietoris, Küneth formula
  - Cohomology
    * Universal Coefficient Theorem
    * Cup products and the cohomology ring
    * Poincaré duality

- **Low-dimensional topology** [3, §§1-3], [10, §§8, 9]
  - Prime decomposition, JSJ decomposition
  - Dehn’s Lemma, Loop and Sphere Theorems
– Heegaard splittings, generalized Heegaard splittings, thin position
– Knot and link surgery, the Lickorish-Wallace theorem, Kirby calculus
– Seifert fibered spaces
– Knot groups and the Wirtinger presentation
– Seifert surfaces

• Mapping Class Groups [2, §§1-5]
  – Computations of $\text{Mod}^+ (\Sigma)$ for $S_{0,2}$, $S_{0,3}$, $A$, $S_{1,0}$, $S_{1,1}$.
  – Dehn twists: including, cutting, and capping.
  – The Birman Exact Sequence
  – Finite generation
  – The lantern relation
  – Finite presentation

REFERENCES