Ch. 12 Probability

12.1 Principles of Counting

Fundamental Principle of Counting
- K selections to be made independently
- \( n_i \) possible choices, \( i = 1, 2, \ldots, K \)

Then total number of choices: \( n_1 \cdot n_2 \cdot \ldots \cdot n_k \)

Ex How many possible CA license plates are there?

Rules: Total of 7 spots
- 1st spot is a 6
- Spots 2-4 are letters and spot 2 and spot 4 cannot be I, O, Q
- Spots 5-7 are digits

e.g. \( 6 \overline{ABC123} \)

\( n_1 = 1 \)
\( n_3 = 26 \)
\( n_2 = 23 = n_4 \)
\( n_5 = n_6 = n_7 = 10 \)

\( \Rightarrow \) # license plates = \( 1 \cdot 23 \cdot 26 \cdot 23 \cdot 10 \cdot 10 \cdot 10 \)
# license plates = 1. 23. 26. 23. 10. 10. 10

\[ \Rightarrow 13,754,000 \]

**Permutations**

- set of ordered arrangements without repetition
  *order matters!*
- # of Permutations: \( P_n = n(n-1)(n-2) \cdots 1 = n! \)

**Ex** Suppose 10 horses are racing. How many outcomes are possible?

\[ 10! \Rightarrow 1^{st} \text{ place: 10 choices} \]
\[ 2^{nd} \text{ place: 9 choices} \]
\[ \vdots \]
\[ 10^{th} \text{ place: 1 choice} \]

\[ \Rightarrow P_n = 10 \cdot 9 \cdot 8 \cdots 1 = 10! \]

**Number of permutations of n objects when k taken**

\[ P_{n,k} = \frac{n!}{(n-k)!} \]

**Ex** Horse racing bets: Exacta: Horses place 1\(^{st}\) and 2\(^{nd}\) in order

Trifecta: Horses place 1\(^{st}\), 2\(^{nd}\), and 3\(^{rd}\) in
Trifecta: Horses place 1st, 2nd, and 3rd in exact order

Superfecta: Horses place 1st, 2nd, 3rd, and 4th in order.

If 7 horses are racing, what are the number of possible Exactas, Trifectas, and Superfectas?

# Exactas: \[ \frac{7!}{(7-2)!} = \frac{7!}{5!} = 7 \cdot 6 = 42 \]

# Trifectas: \[ \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210 \]

# Superfectas: \[ \frac{7!}{(7-4)!} = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840 \]

**Combinations**

- set of objects that is unordered, without repetition

# of combinations of n objects when taking k:

\[ C_{n,k} = \frac{n!}{k!(n-k)!} = \binom{n}{k} \] "n choose k"

**Ex** CA Mega Millions: $266 Million estimated jackpot

How many possible Jackpot options?

Rules: 5 numbers from 1 to 70, order
Rules:

- 5 numbers from 1 to 70, order does not matter
- 1 number from 1 to 25

1st 5 numbers: \( C_{70,5} = \frac{70!}{5!(70-5)!} \)

\[ = \frac{70!}{5! \cdot 65!} \]

\[ = \frac{70 \cdot 69 \cdot 68 \cdot 67 \cdot 66}{5!} \]

\[ = 12,103,014 \]

Total number of choices when buying a lotto ticket is:

\[ 12,103,014 \times 25 = 302,575,350 \]

12.2 What is Probability?

- Given an experiment with distinct possible outcomes
  - Subset of the sample space is called an event
Manipulating sets

- Subset of the sample space
- Complement of $E$, $E^c$
- Union of $E$ and $F$, $E \cup F$
- Intersection of $E$ and $F$, $E \cap F$
- $E$ and $F$ are disjoint: $E \cap F = \emptyset$

$E \cup F = F \cup E$
Commutative: \( EUF = FUE \)

Associative: \( (EUF) U G = EU (FUG) \)

Distributive: \( (EUF) \cap G = (E \cap G) U (F \cap G) \)

Ex

An experiment is tossing a coin and drawing a card from a deck.

a) How many elements are in the sample space?

b) How many elements in the event "heads and an ace"

c) How many elements in the event "tails and a face"

d) List the elements in "heads and spade"

\[ \text{Flip a coin and draw a card} \]

\[ \text{2-sided coin} \]

\[ \text{52 cards} \]

\[ a) \ 2 \cdot 52 = 104 \text{ elements in sample space} \]

\[ b) \ "\text{heads and an ace}" \]

\[ \frac{1}{4} \]
b) \( \frac{\text{heads}}{4} = 1 \) \[1 \cdot 4 = 4\]

c) \( \frac{\text{tails and a face}}{1} \)

\[
\begin{align*}
\text{4 suits} & \quad \text{3 faces per suit} \\
\text{12 face cards} &
\end{align*}
\]

\[12 \cdot 1 = 12\]

d) \( \frac{\text{heads and spades}}{1} \)

\[
\begin{align*}
\text{Ace} & \\
\text{2} & \\
\text{3} & \\
\text{5} & \\
\text{Q} & \\
\text{K} &
\end{align*}
\]

\[1 \cdot 13 = 13\]