Discussion 5/21/2020

Thursday, May 21, 2020 5:05 PM

Midterm II: Same format as Midterm I

<u>Ex</u> Show that $x_1 = e^t \cos(2\sqrt{2}t)$ and $x_2 = -\sqrt{2}e^t \sin(2\sqrt{2}t)$ are solutions to

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix}.$$

1st let's compute $\frac{d\vec{x}}{dt}$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} e^t\cos(2\sqrt{2}t) - 2\sqrt{2}e^t\sin(2\sqrt{2}t) \\ -\sqrt{2}e^t\sin(2\sqrt{2}t) - 4e^t\cos(2\sqrt{2}t) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}x_1 \\ \frac{d}{dt}x_2 \end{bmatrix}$$

2nd let's compute $A\vec{x}$

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} e^t\cos(2\sqrt{2}t) \\ -\sqrt{2}e^t\sin(2\sqrt{2}t) \end{bmatrix}$$

$$= \begin{bmatrix} e^t\cos(2\sqrt{2}t) - 2\sqrt{2}e^t\sin(2\sqrt{2}t) \\ -4e^t\cos(2\sqrt{2}t) - \sqrt{2}e^t\sin(2\sqrt{2}t) \end{bmatrix}$$

$\rightsquigarrow \frac{d\vec{x}}{dt} = A\vec{x}$, so x_1, x_2 are solutions to $\frac{d\vec{x}}{dt} = A\vec{x}$.

<u>Solving linear systems of differential equations</u>

1. Find eigenvalues, eigenvectors
2. General Solution
3. [illegible] conditions

2. General solution

3. Use initial conditions

<u>Ex</u> Find the solution to the IVP

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

1. Eigenvalues : $\lambda_1 = 3, \ \lambda_2 = 1$

$\det\left(\begin{bmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}\right) = 0$

$(3-\lambda)(1-\lambda) = 0$

$\lambda = 3, \ \lambda = 1$

Eigenvectors: $\underline{\lambda_1 = 3}$

$$\begin{bmatrix} 3-3 & 0 \\ 0 & 1-3 \end{bmatrix} \vec{v}_1 = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \vec{v}_1 = 0$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$

$-2v_{12} = 0$

$v_{12} = 0$

v_{11} is "free"

$\vec{v}_1 = \begin{bmatrix} v_{11} \\ 0 \end{bmatrix} = v_{11} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$

$\underline{\lambda_2 = 1}$

$$\begin{bmatrix} 3-1 & 0 \\ 0 & 1-1 \end{bmatrix} \vec{v}_2 = 0$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \vec{v}_2 = 0$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. General Solution:

$$\vec{x} = c_1 e^{\overset{\lambda_1}{3t}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{3t} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3. Use initial conditions

$$\vec{x}(0) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

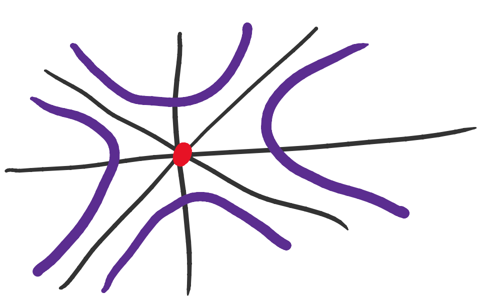
$$\Rightarrow c_1 = -2, \quad c_2 = 4$$

$$\vec{x}(t) = -2e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

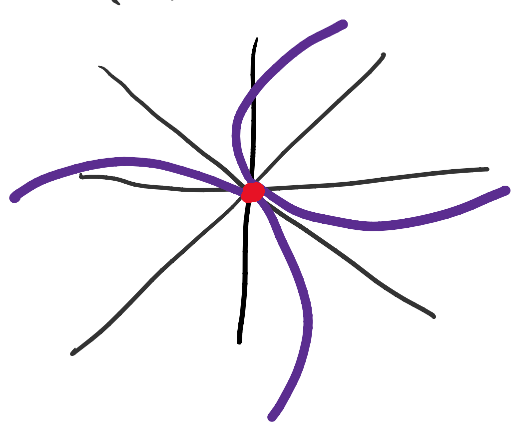
$$= \begin{bmatrix} -2e^{3t} \\ 4e^{t} \end{bmatrix}$$

<u>Types of Equilibria</u>:

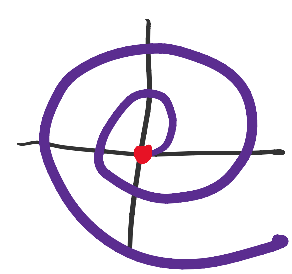
Saddles
($\lambda_1 \cdot \lambda_2 < 0$)



Nodes
($\lambda_1 \lambda_2 > 0$)



Spirals
(λ complex
$Re(\lambda) \neq 0$)



Sketching the Phase Plane:

Sketching the Phase Plane:

1. Plot eigenvalues
2. Plot null clines
3. Consider the behavior of equilibria
4. Draw solution curves

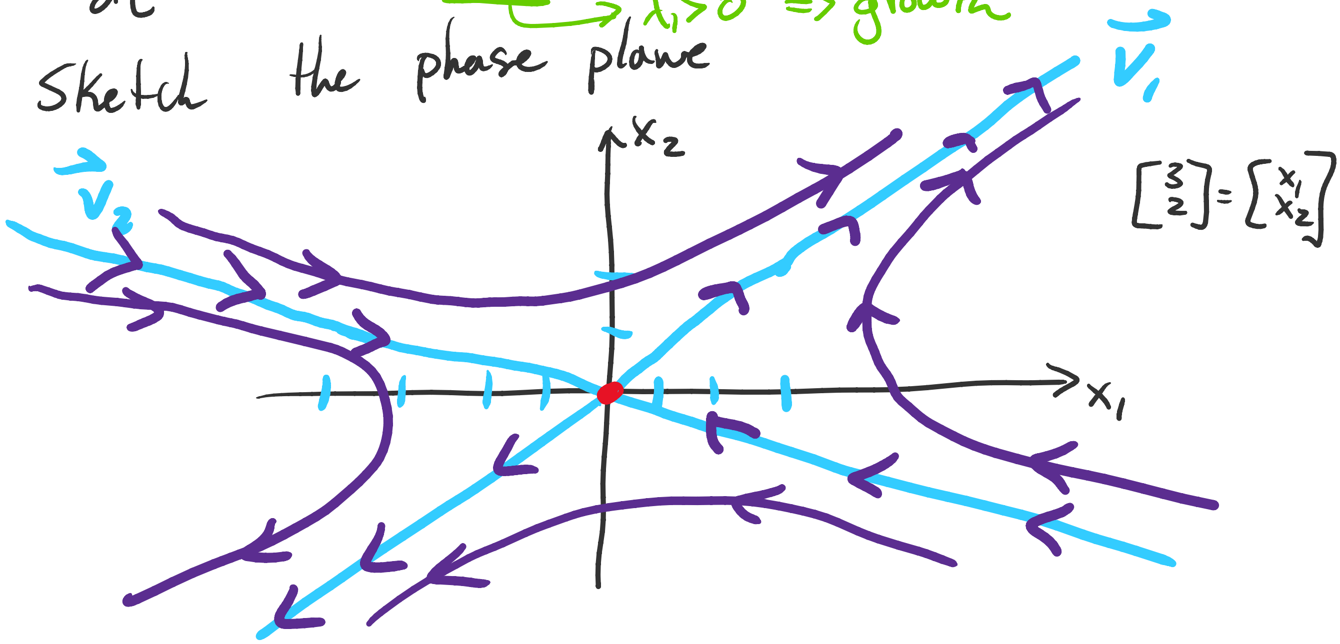
$e^{\lambda t}$ }

$\lambda_2 < 0 \Rightarrow$ decay

Ex $\frac{d\vec{x}}{dt} = A\vec{x}$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\lambda_1 > 0 \Rightarrow$ growth

Sketch the phase plane



Ex Construct the phase plane. What does the equilibrium look like?

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \vec{x}$$

Eigenvalues: $\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0$

Eigenvalues: $\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0$

$$(1-\lambda)(-1-\lambda) - 4 = 0$$

$$\lambda^2 - 5 = 0$$

$$\lambda = \pm\sqrt{5}$$

Eigenvectors: $\underline{\lambda_1 = \sqrt{5}}$ $\qquad \vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$

$$\begin{bmatrix} 1-\sqrt{5} & 2 \\ 2 & -1-\sqrt{5} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

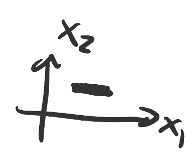
$$(1-\sqrt{5})v_{11} + 2v_{12} = 0$$

$$v_{12} = -\frac{(1-\sqrt{5})}{2} v_{11}$$

$$\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \vec{v}_1 = \begin{bmatrix} v_{11} \\ -\frac{(1-\sqrt{5})}{2} v_{11} \end{bmatrix} = v_{11} \underbrace{\begin{bmatrix} 1 \\ -\frac{(1-\sqrt{5})}{2} \end{bmatrix}}$$

Take $v_{11} = 1$ $\qquad \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}$

$\rightsquigarrow$ can find that $\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{(1+\sqrt{5})}{2} \end{bmatrix}$



2. Nullclines

$\underline{x_1' = 0}$ (vertical lines)

$$x_1' = x_1 + 2x_2$$

$$x_1' = 0 = x_1 + 2x_2$$

$$x_2 = -\frac{1}{2}x_1$$

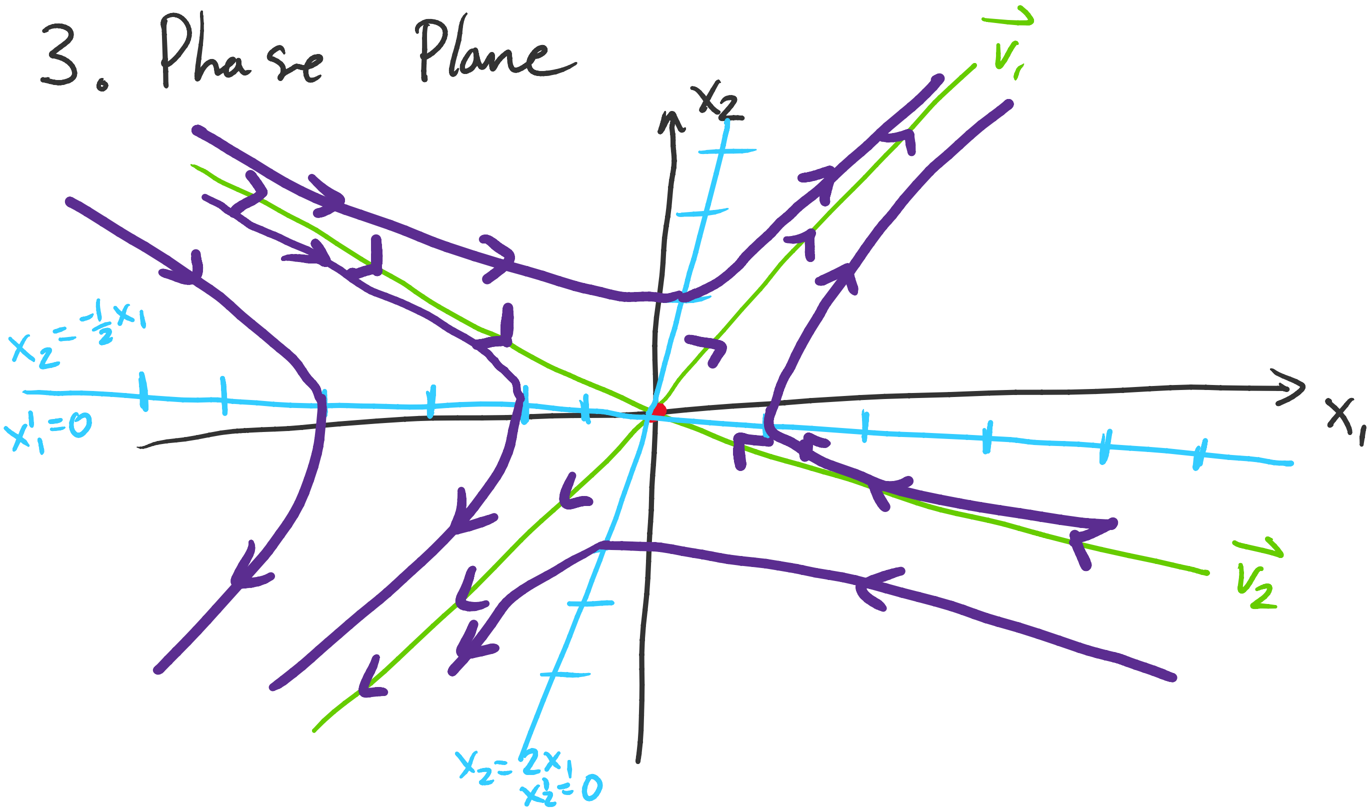
$\underline{x_2' = 0}$ (horizontal lines)

$$x_2' = 2x_1 - x_2$$

$$0 = 2x_1 - x_2$$

$$x_2 = 2x_1$$

3. Phase Plane



Counting

- combinations - order does not matter
- permutations - order matters

Ex A jar contains 7 different colored balls
How many sequences of 5 colors
possible?

$$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = \binom{7}{1}\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}$$

How many color combinations

$$\binom{7}{5} = \frac{7!}{5!\,2!} = 21$$

→ balls one at a time,

If you remove 2 balls, one at a time, how many color sequences possible?

$$\binom{7}{1}\binom{6}{1} = 7 \cdot 6 = 42$$

How many color combinations?

$$\binom{7}{2} = \frac{7!}{2! \cdot 5!} = 21$$

4 ways to draw ace
52 total cards

Then, $P(\text{drawing ace}) = \frac{4}{52}$

Calc. Room 6 7