Probability

- Recall we have some experiment, with distinct possible outcomes, sample space, \( \Omega \).

Equally likely outcomes, \( E \) event

\[
P(E) = \frac{n(E)}{n(\Omega)}
\]

\( 0 \leq P(E) \leq 1, \quad P(\Omega) = 1 \)

Probability of mutually exclusive events, \( E, F \)

\[
P(E \cup F) = P(E) + P(F)
\]

Complement rule: \( P(E^c) = 1 - P(E) \)

Union rule: \( P(E \cup F) = P(E) + P(F) - P(E \cap F) \)
Here, E and F are any two events.

\[ \text{Ex} \] A die is rolled and we observe the number. Are the Events E and F mutually exclusive? Find \( P(E \cup F) \)

a) \( E \): The number is even
b) \( F \): The number is odd

\[ E \cap F = \{ \text{number is even and odd} \} = \emptyset \]

\( \implies E, F \) are mutually exclusive, so

\[ P(E \cup F) = P(E) + P(F) = \frac{1}{2} + \frac{1}{2} = 1 \]
b) $E$: The number is even

$F$: The number is greater than 4

$\text{ENF} = \{ \text{number is even and greater than 4} \}$

$= \{ 6 \}$

$\Rightarrow E$ and $F$ are not mutually exclusive

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6}$

$= \frac{4}{6}$
Conditional Probability

- we restrict the sample space given some condition

"Given _, what is the probability of _?"

Given $F$, probability of $E$?

Conditional probability of $E$ given $F$ is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$
Complement rule of Conditional Prob.

\[ P(E^c | F) = 1 - P(E | F) \]

Ex. Two dice are rolled.

a) What is the probability that the sum of the numbers is greater than 8?

For sum to be greater than 8

\[
\begin{array}{c|cccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & none & none & 6 & 5, 6 & 4, 5, 6 \\
3 & none & none & 6 & 5, 6 & 3, 4, 5, 6 \\
4 & none & none & 6 & 5, 6 & 3, 4, 5, 6 \\
5 & none & none & 6 & 5, 6 & 3, 4, 5, 6 \\
6 & none & none & 6 & 5, 6 & 3, 4, 5, 6 \\
\end{array}
\]

\[ \Rightarrow P(\text{sum} > 8) = \frac{10}{6 \cdot 6} = \frac{10}{36} \]
6) What is the probability that the sum is greater than 8 given that the first die shows a 3?

E = \{ sum > 8 \}  
E^c = \{ sum \leq 8 \}  
P(E|F) = \frac{P(E \cap F)}{P(F)}

= \frac{\frac{1}{36}}{\frac{16}{36}}

= \frac{1}{6}

1^{st} die must be a 3  
2^{nd} die can be 1, 2, \ldots, 6

\( \) What is the probability that the first die shows a 3, given
that the sum is greater than 8?

\[ P(F \mid E) = \frac{P(E \cap F)}{P(E)} \]

\[ = \frac{1}{36} \]

\[ = \frac{10}{36} \]

\[ = \frac{1}{10} \]

**Independent Events**

\[ P(E \cap F) = P(E) \cdot P(F) \]

- Can extend this to more than 2 events
Ex. We roll a die twice. Let

\[ E = \{ \text{1st roll is a 6}\} \]
\[ F = \{ \text{2nd roll is a 6}\} \]

a) Find the probability of showing a 6 on both rolls.

Only 1 way to roll 2 6's
36 possible outcomes

\[ P(E \cap F) = \frac{1}{36} \]

b) Are the events E and F independent?

\[ P(E) = \frac{1}{6}, \quad P(F) = \frac{1}{6} \]

\[ P(E \cap F) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(E) \cdot P(F) \]
\[ \Rightarrow P(E \cap F) = P(E)P(F) \]

\[ \Rightarrow E \text{ and } F \text{ are independent events.} \]

- Let \( A, B \) be two sets
  - \( A \) and \( B \) are disjoint if \( A \cap B = \emptyset \)

\( A \cap B \)

- Let \( A \) and \( B \) be two events
  - If \( P(A \cap B) = P(A)P(B) \), then \( A \) and \( B \) are independent
  - If \( A \) and \( B \) are disjoint events then we say they are mutually exclusive.
Law of Total Probability

\[ S_2 \]

\[ F_1, F_2, \ldots, F_6 \] are pairwise disjoint, exhaustive events

i.e. \( F_1 \cap F_2 \cap \ldots \cap F_6 = \emptyset \)  

\[ F_1 \cup F_2 \cup \ldots \cup F_6 = \Omega \]  

Let \( E \) be any event, and \( F_1, F_2, \ldots, F_k \) a collection of pairwise disjoint, exhaustive events. Then
\[ P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \ldots + P(E|F_k)P(F_k) \]
\[ = \sum_{i=1}^{K} P(E|F_i)P(F_i) \]

**Ex.** One card randomly removed from a standard deck. If a 2nd card is removed, what is the probability that the second card is a face card?

\[ E = \{ \text{2nd card a face} \} \]
\[ \{ \text{1st card is a face} \} \cup \{ \text{1st card not a face} \} \]
\[ = \Omega \]
\[ F_1 = \text{1st card a face} \]
\[ F_2 = \text{1st card not a face} \]

\[
P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2)
\]
\[
= \frac{11}{51} \cdot \frac{12}{52} + \frac{12}{51} \cdot \left( 1 - \frac{12}{52} \right)
\]
\[
= \frac{11 \cdot 12 + 12 \cdot 40}{51 \cdot 52}
\]
\[
= \frac{51 \cdot 12}{51 \cdot 52}
\]
\[
= \frac{3}{13}
\]
\[ P(A \cap B \cap C) = P(A) P(B) P(C) \]