Probability

- recall we have some experiment, with $\underbrace{\text{distinct possible outcomes}}_{\text{sample space, } \Omega}$

Equally likely outcomes, E event

$$P(E) = \frac{n(E)}{n(\Omega)}$$

$$0 \le P(E) \le 1, \quad P(\Omega) = 1$$

Probability of Mutually exclusive events, E, F

$$P(E \cup F) = P(E) + P(F)$$

Complement rule: $P(E^c) = 1 - P(E)$

Union rule: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

- Here, E and F are any two events

<u>Ex</u> A die is rolled and we observe the number. Are the Events E and F mutually exclusive? Find $P(E \cup F)$

a) E: The number is even

F: The number is odd

$E \cap F = \{\text{number is even and odd}\}$

$= \emptyset$

$\Rightarrow E, F$ are mutualy exclusive, so

$$P(E \cup F) = P(E) + P(F)$$
$$= \frac{1}{2} + \frac{1}{2} = 1$$

b) E: The number is even

F: The number is greater than 4

$E \cap F = \{$number is even and greater than 4$\}$

$= \{6\}$

$\Rightarrow$ E and F are not mutually exclusive

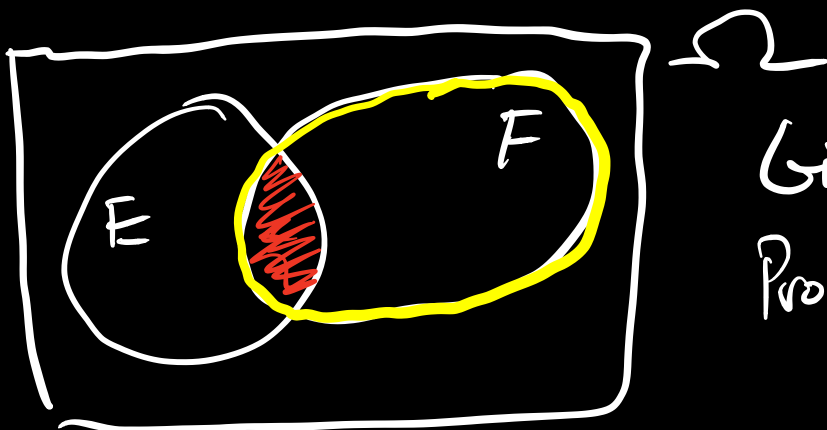
$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6}$$

$$= \frac{4}{6}$$

Conditional Probability

- we restrict the sample space given some condition
  "Given ___, what is the probability of ___?"



Given F, Probability of E?

Conditional probability of E given F is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Complement rule of Conditional Prob.

$$P(E^c | F) = 1 - P(E|F)$$

<u>Ex</u> Two dice are rolled.

a) What is the probability that the sum of the numbers is greater than 8?

For sum to be greater than 8

1st die	2nd die
1	none
2	none
3	6
4	5, 6
5	4, 5, 6
6	3, 4, 5, 6

$$\Rightarrow P(\text{sum} > 8) = \frac{10}{6 \cdot 6} = \frac{10}{36}$$

b) What is the probability that the sum is greater than 8 given that the first die shows a 3?

$E = \{\text{sum} > 8\}$

$F = \{1^{st} \text{ die a } 3\}$

$E^c = \{\text{sum} \leq 8\}$

$P(E^c | F)$

$= 1 - P(E|F)$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{1/36}{1 \cdot 6/36}$$

1^{st} die must be a 3

2^{nd} die can be 1, 2, ..., or 6

$$= \frac{1}{6}$$

c) What is the probability that the first die shows a 3, given that the sum is greater than 8?

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

$$= \frac{\frac{1}{36}}{\frac{10}{36}}$$

$$= \frac{1}{10}$$

Independent Events

$$P(E \cap F) = P(E)P(F)$$

- Can extend this to more than 2 events

Ex We roll a die twice. Let

$E = \{1^{st} \text{ roll is a } 6\}$

$F = \{2^{nd} \text{ roll is a } 6\}$

a) Find the probability of showing a 6 on both rolls.

Only 1 way to roll 2 6's

36 possible outcomes

$$P(E \cap F) = \frac{1}{36}$$

b) Are the events E and F independent?

$$P(E) = \frac{1}{6}, \quad P(F) = \frac{1}{6}$$

$$P(E \cap F) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(E) \cdot P(F)$$

$\Rightarrow P(E \cap F) = P(E)P(F)$

$\Rightarrow E$ and F are independent events.

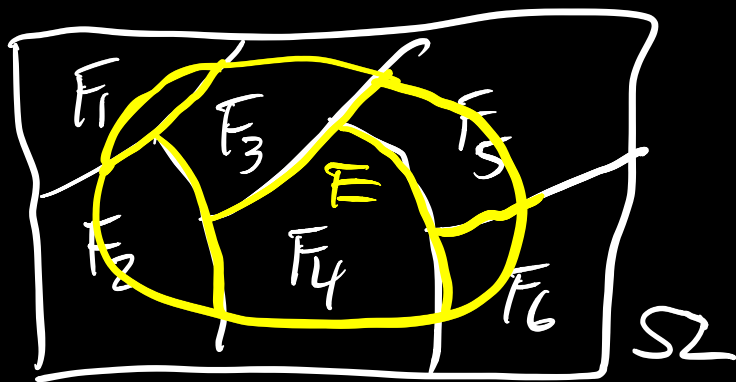
- Let A, B be two sets. A and B are disjoint if

$$A \cap B = \emptyset$$

Let A and B be two events

- If $P(A \cap B) = P(A)P(B)$, then A and B are independent
- If A and B are disjoint events then we say they are mutually exclusive.

Law of Total Probability



$F_1, F_2, ---, F_6$ are pairwise disjoint, exhaustive events

$F_i \cap F_j = \emptyset$

i.e. $F_1 \cap F_2 \cap --- \cap F_6 = \emptyset$ } pairwise disjoint

exhaustive { $F_1 \cup F_2 \cup --- \cup F_6 = \Omega$

Let E be any event, and $F_1, F_2, ---, F_K$ a collection of pairwise disjoint, exhaustive events.

Then

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \cdots + P(E|F_K)P(F_K)$$
$$= \sum_{i=1}^{K} P(E|F_i)P(F_i)$$

Ex One card randomly removed from a standard deck. If a 2nd card is removed, what is the probability that the second card is a face card?

$E = \{2^{nd} \text{ card a face}\}$

$\{1^{st} \text{ card is a face}\} \cup \{1^{st} \text{ card not a face}\}$
$= \Omega$

$F_1 = \{1^{st} \text{ card a face}\}$

$F_2 = \{1^{st} \text{ card not a face}\}$

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2)$$

$$= \frac{11}{51} \cdot \frac{12}{52} + \frac{12}{51} \cdot \underbrace{\left(1 - \frac{12}{52}\right)}_{\frac{52-12}{52}}$$

$$= \frac{11 \cdot 12 + 12 \cdot 40}{51 \cdot 52}$$

$$= \frac{51 \cdot 12}{51 \cdot 52}$$

$$= \frac{3}{13}$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$