Probability

- recall we have some experiment, with distinct possible outcomes
  ↳ Ω sample space

Equally likely outcomes

E - event

$$P(E) = \frac{n(E)}{n(\Omega)}$$

$0 \leq P(E) \leq 1$, $P(\Omega) = 1$

Probability of Mutually Exclusive events, E, F

$$P(E \cup F) = P(E) + P(F)$$

Complement rule: $P(E^c) = 1 - P(E)$

Union rule: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

- Here, E and F are any two events

Ex A die is rolled and we observe the number. Are the events E and F mutually exclusive? Find $P(E \cup F)$

a) E: The number is even

F: The number is odd.

$E \cap F = \{$number is even and odd$\}$

$= \varnothing$

$\Rightarrow$ E and F are mutually exclusive, so

$$P(E \cup F) = P(E) + P(F)$$
$$= \frac{3}{6} + \frac{3}{6}$$
$$= 1$$

b) E: The number is even

F: The number is greater than 4

$E \cap F = \{$number is even and greater than $4\}$

$= \{6\}$ $\quad 1 = n(E \cap F)$

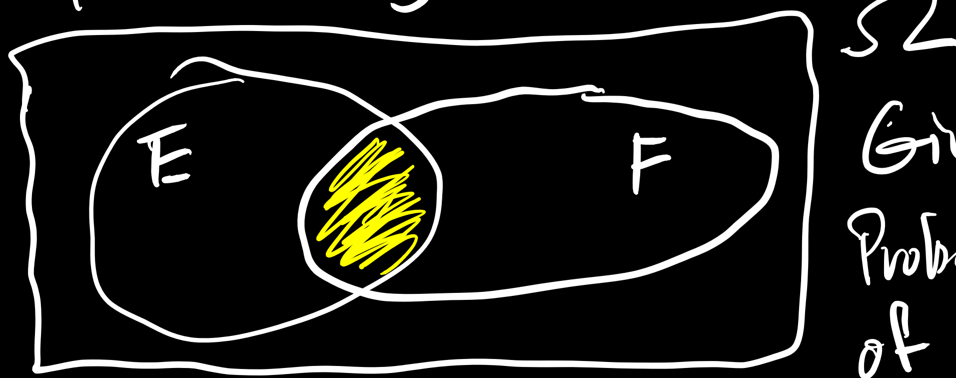
$\Rightarrow$ E and F are not mutually exclusive, so

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$= \frac{3}{6} + \frac{2}{6} - \underbrace{\frac{1}{6}}_{\frac{n(E\cap F)}{n(\Omega)}}$$
$$= \frac{4}{6}$$

Conditional Probability

- restrict the sample space given some condition

"Given ____, what is the probability of ____?"



Given F, Probability of E?

Conditional probability of E given F is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

- complement rule for conditional probability

$$P(E^c|F) = 1 - P(E|F)$$

Ex We roll two dice.

a) What is the probability that the sum of the numbers is greater than 8?

For sum to be greater than 8:

1st die	2nd die
1	none
2	none
3	6
4	5, 6
5	4, 5, 6
6	3, 4, 5, 6

$$P(\text{sum} > 8) = \frac{10}{\underbrace{6 \cdot 6}} = \frac{10}{36}$$

6 possible rolls for 1st die
6 possible rolls for 2nd die

b) What is the probability that the sum is greater than 8 given that the first die shows a 3?

$E = \{\text{sum} > 8\}$

$F = \{1^{st} \text{ die is a } 3\}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{1/36}{\frac{1 \cdot 6}{36}} = \frac{1}{6}$$

c) What is the probability that the 1^{st} die shows a 3, given that the sum is greater than 8?

$$P(F|E) = \frac{P(F \cap E)}{P(E)}$$

$$= \frac{1/36}{10/36}$$

$$= \frac{1}{10}$$

Independent Events

$$P(E \cap F) = P(E)P(F)$$

- can be extended to more than 2 events

Ex We roll a die twice. Let

$E = \{1^{st}$ roll is a $6\}$

$F = \{2^{nd}$ roll is a $6\}$

a) Find the probability of showing a 6 on both rolls

Only 1 way to roll (6,6)
36 possible rolls

$$P(E \cap F) = \frac{1}{36}$$

b) Are the events E and F independent?

$$P(E) = \frac{1}{6}, \quad P(F) = \frac{1}{6}$$

$$P(E \cap F) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(E)P(F)$$

$$\Rightarrow P(E \cap F) = P(E)P(F)$$

$\Rightarrow$ E and F are independent

Law of Total Probability

$F_1, \ldots, F_6$ pairwise disjoint, exhaustive events

i.e. $F_1 \cap F_2 \cap \cdots \cap F_6 = \emptyset$ and

$F_1 \cup F_2 \cup \cdots \cup F_6 = \Omega$

Let E be any event, and $F_1, \ldots, F_K$ be a collection of pairwise disjoint, exhaustive events. Then,

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \cdots + P(E|F_K)P(F_K)$$

$$= \sum_{i=1}^{K} P(E \mid F_i)\, P(F_i)$$

Ex One card is randomly removed from a standard deck. If a 2^{nd} card is removed, what is the probability that the 2^{nd} card is a face.

$E = \{2^{nd}$ card a face$\}$

$\{1^{st}$ card is a face$\} \cup \{1^{st}$ card not a face$\}$

$= \Omega$

$F_1 = \{1^{st}$ card face$\}$

$F_2 = \{1^{st}$ card not a face$\}$

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2)$$

$$= \frac{11}{51}\cdot\frac{12}{52} + \frac{12}{51}\cdot\underbrace{\left(1-\frac{12}{52}\right)}_{\frac{52-12}{52}}$$

$$= \frac{11\cdot 12 + 12(40)}{51\cdot 52}$$

$$= \frac{51\cdot 12}{51\cdot 52}$$

$$= \frac{12}{52} = \frac{3}{13}$$

<u>Baye's Rule</u>

Events E and F

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E)}$$