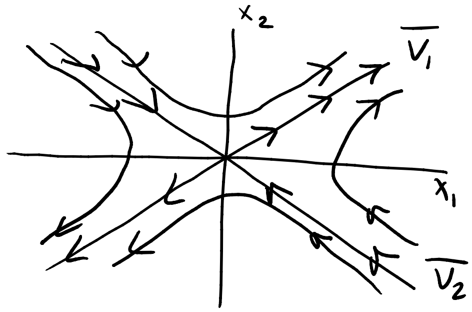
Discussion 5/6/2020

Thursday, May 7, 2020 4:02 PM

Qualitative Behavior of Systems of DE

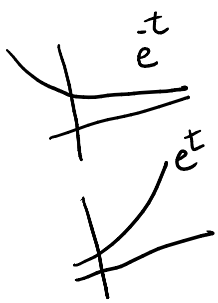
The behavior of a system of DE is characterized by the eigenvalues

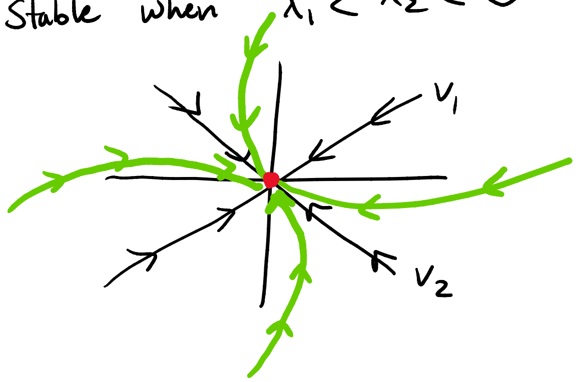
- Saddles, λ_1, λ_2 are real, $\lambda_1 > 0$, $\lambda_2 < 0$



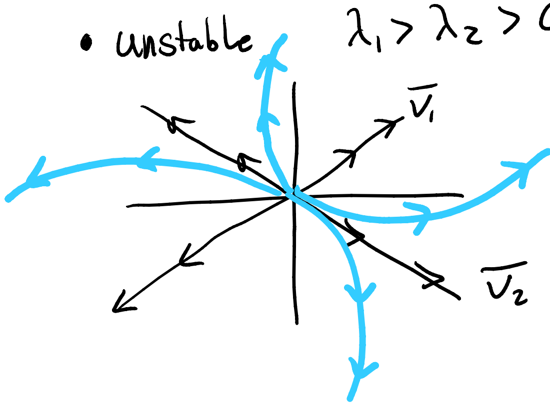
- Nodes, λ_1, λ_2 are real
  - Stable when $\lambda_1 < \lambda_2 < 0$

$\bar{x} = c_1 e^{\lambda_1 t}\, \bar{v}_1 + c_2 e^{\lambda_2 t}\, \bar{v}_2$

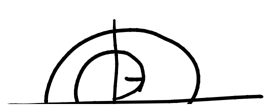




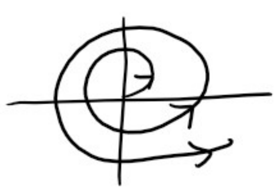
  - unstable $\lambda_1 > \lambda_2 > 0$



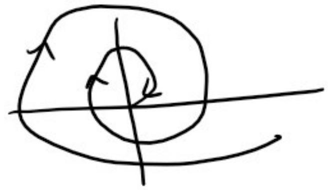
- Spirals, complex eigenvalues $\lambda = a \pm ib$
  - $a > 0$ unstable



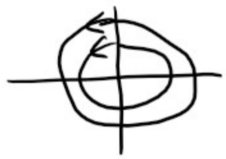
- $a > 0$
  unstable



- $a < 0$
  stable



- $a = 0$



- choose a test point to determine direction of rotation (CW or CCW)

Long-term behavior

- origin is stable if and only if $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$
  if and only if $\det A > 0$ and $\mathrm{trace}(A) < 0$

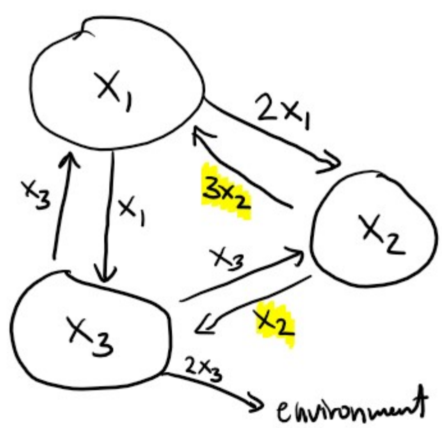
Compartment models

- how do we form a DE to model some system?

$$\frac{dx_1}{dt} = \underline{-3x_1} + 3x_2 + x_3$$

$$\frac{dx_2}{dt} = 2x_1 \underline{- 4x_2} + x_3$$

$$\frac{dx_3}{dt} = x_1 + x_2 \underline{- 4x_3}$$



- "where is stuff flowing"

10.4 Systems of Nonlinear DEs

- previously had linear systems of DEs

$$\frac{d\bar{x}}{dt} = A\bar{x}$$

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$$\frac{d\bar{x}}{dt} = A\bar{x}$$

Now, we want to study more complicated models, so we have a more general system of autonomous nonlinear DE

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

where f_1 and f_2 are arbitrary functions with continuous first partial derivatives

Ex (Predator-Prey)

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} \alpha x - \beta xy \\ \delta xy - \gamma y \end{bmatrix}$$

- Difficult or impossible to solve nonlinear systems, so we look at qualitative aspects of the system.

Local Stability Analysis

- use the linearization to determine local behavior of equilibria

1. Find equilibria
2. Compute Jacobian

$$J(x_1, x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$$

3. Compute eigenvalues of the Jacobian evaluated at equilibria
4. Classify equilibria.

i) λ_1, λ_2 real, nonzero

- $\lambda_1 \cdot \lambda_2 > 0$ (node) $\rightarrow$ $\lambda_1 + \lambda_2 > 0$ unstable; $\lambda_1 + \lambda_2 < 0$ stable
- $\lambda_1 \cdot \lambda_2 < 0$ $\rightarrow$ Saddle point (unstable)

ii) λ_1, λ_2 complex conjugates

$\bullet\ \lambda_1 \ \ldots$

ii) λ_1, λ_2 complex conjugates

- $Re(\lambda_1) > 0 \to$ unstable spiral
- $Re(\lambda_1) < 0 \to$ Stable spiral

iii) Inconclusive otherwise.

__Ex__ Find and classify the equilibria

$$\frac{dx_1}{dt} = x_1 - 2x_1^2 - 6x_1x_2$$

$$\frac{dx_2}{dt} = 2x_2 - 8x_2^2 - 2x_1x_2$$

1. Find equilibria

$\frac{dx_1}{dt} = 0$

$x_1 - 2x_1^2 - 6x_1x_2 = 0$

$x_1(1 - 2x_1 - 6x_2) = 0$

$x_1 = 0 \qquad 1 - 2x_1 - 6x_2 = 0$

$\frac{dx_2}{dt} = 0$

$2x_2 - 8x_2^2 - 2x_1x_2 = 0$

$x_2(2 - 8x_2 - 2x_1) = 0$

$x_2 = 0 \qquad 2 - 8x_2 - 2x_1 = 0$

__$x_1 = 0$__

- $x_2 = 0$
- $2 - 8x_2 - \cancelto{0}{2x_1} = 0 \Rightarrow x_2 = \frac{1}{4}$

__$x_2 = 0$__

- $x_1 = 0$
- $1 - 2x_1 - \cancelto{0}{6x_2} = 0 \Rightarrow x_1 = \frac{1}{2}$

__$1 - 2x_1 - 6x_2 = 0$__ $\to 2x_1 = 1 - 6x_2$

- $2 - 8x_2 - 2x_1 = 0$

$2 - 8x_2 - 1 + 6x_2 = 0$

$-2x_2 = -1$

$x_2 = 1/2$

$2x_1 = 1 - 6(\frac{1}{2})$

$x_1 = -1$

Equilibria: $(0,0), (0,\frac{1}{4}), (\frac{1}{2},0), (-1,\frac{1}{2})$

2. Compute Jacobian

$$J(x_1, x_2) = \begin{pmatrix} 1-4x_1-6x_2 & -6x_1 \\ -2x_2 & 2-16x_2-2x_1 \end{pmatrix}$$

3. Eigenvalues (a) b) (c)

$$J(0,0) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \quad (1-\lambda)(2-\lambda)=0$$

$\lambda_1 = 1, \ \lambda_2 = 2$

$$J(\tfrac{1}{2}, 0) = \begin{pmatrix} 1-2 & -3 \\ 0 & 1 \end{pmatrix}$$

$\lambda_1 = -1, \ \lambda_2 = 1$

$(-\tfrac{1}{2}-\lambda)(-2-\lambda)=0$

$$J(0, \tfrac{1}{4}) = \begin{pmatrix} 1-\tfrac{3}{2}-\lambda & 0 \\ -\tfrac{1}{2} & -2-\lambda \end{pmatrix}$$

$\lambda_1 = -\tfrac{1}{2}, \ \lambda_2 = -2$

$$J(-1, \tfrac{1}{2}) = \begin{pmatrix} 2 & 6 \\ -1 & -4 \end{pmatrix}$$

$$0 = \det(J(-1,\tfrac{1}{2}) - \lambda I) = (2-\lambda)(-4-\lambda)+6$$
$$= \lambda^2 + 2\lambda - 2$$
$$\lambda = \frac{-2 \pm \sqrt{4+8}}{2}$$
$$= -1 \pm \sqrt{3}$$

$\lambda_1 > 0, \ \lambda_2 < 0$

4. Classification of Equilibria

$(0,0)$ $\lambda_1 \cdot \lambda_2 > 0$, $\lambda_1 + \lambda_2 > 0$ $\rightarrow$ unstable node

$(0,\tfrac{1}{4})$ $\lambda_1 \cdot \lambda_2 > 0$, $\lambda_1 + \lambda_2 < 0$ $\rightarrow$ stable node

$(\tfrac{1}{2},0)$ $\lambda_1 \cdot \lambda_2 < 0$ $\rightarrow$ saddle point

$(-1,\tfrac{1}{2})$ $\lambda_1 \cdot \lambda_2 < 0$ $\rightarrow$ Saddle point