Discussion 5/7/2020

Thursday, May 7, 2020 4:01 PM

Last time: We ended with going over different types of equilibria (stable/unstable nodes/spirals, saddle points)

Ex Sketch the nullclines, eigenvectors, and solution curves of the system

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} 2 & 0 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \lambda_1 = 2 \quad \bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \lambda_2 = 4 \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Nullclines: $x_1' = 0$ (vertical change) $\qquad x_2' = 0$ (horizontal change)

$2x_1 = 0 \qquad\qquad -6x_1 + 4x_2 = 0$

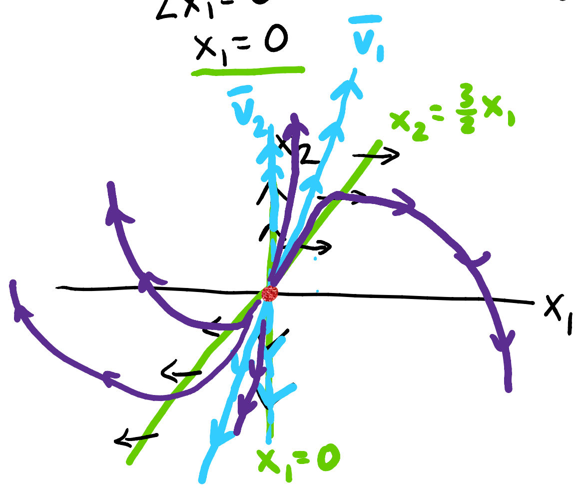
$x_1 = 0 \qquad\qquad x_2 = \frac{3}{2}x_1$

$x_1 = 0$

$x_2' = 4x_2$

$x_2 = \frac{3}{2}x_1$

$x_1' = 2x_1$



Long-term behavior

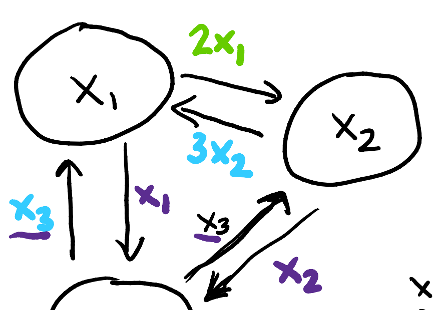
- origin is stable if and only if $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$ if and only if $\det A > 0$ and $trace(A) < 0$.

Compartment models

- how do we form a differential equation to model some system?

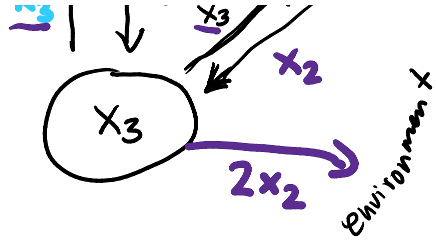
$$\frac{dx_1}{dt} = -3x_1 + 3x_2 + x_3$$

$$\frac{dx_2}{dt} = 2x_1 - 4x_2 + x_3$$



$$\frac{dx_2}{dt} = 2x_1 - 4x_2 + x_3$$

$$\frac{dx_3}{dt} = x_1 + x_2 - 4x_3$$



- "where is stuff flowing"

10.4 Systems of Nonlinear DE

- previously considered linear systems of the form

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

Now we want to consider more complicated models, so we have a more general system of autonomous nonlinear DE:

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

where f_1 and f_2 are arbitrary functions with continuous first partial derivatives

Ex (Predator Prey)

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} \alpha x - \beta xy \\ \delta xy - \gamma y \end{bmatrix}$$

- difficult or impossible to solve, so we look at the qualitative aspects of the system.

Local Stability Analysis

- use the linearization to determine local behavior of equilibria
  1. Find equilibria
  2. ... Jacobian:

1. Find equilibria
2. Compute Jacobian:

$$J(x_1, x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

3. Compute eigenvalues of the Jacobian at equilibria

4. Classify equilibria:

i) λ_1, λ_2 are real, nonzero

- $\lambda_1 \cdot \lambda_2 > 0$ (node) $\rightarrow$ $\lambda_1 + \lambda_2 > 0$ unstable; $\rightarrow$ $\lambda_1 + \lambda_2 < 0$ stable
- $\lambda_1 \cdot \lambda_2 < 0$ $\rightarrow$ Saddle point (unstable)

ii) λ_1, λ_2 are complex conjugates

- $Re(\lambda_1) > 0$ $\rightarrow$ unstable spiral
- $Re(\lambda_1) < 0$ $\rightarrow$ stable spiral

iii) Inconclusive otherwise

<u>Ex</u> Find and classify the equilibria

$$\frac{dx_1}{dt} = x_1 - 2x_1^2 - 6x_1x_2$$

$$\frac{dx_2}{dt} = 2x_2 - 8x_2^2 - 2x_1x_2$$

1. Find equilibria

$$\frac{dx_1}{dt} = 0 \qquad\qquad \frac{dx_2}{dt} = 0$$

$$x_1 - 2x_1^2 - 6x_1x_2 = 0 \qquad\qquad 2x_2 - 8x_2^2 - 2x_1x_2 = 0$$

dt

$x_1 - 2x_1^2 - 6x_1x_2 = 0$

$x_1(1 - 2x_1 - 6x_2) = 0$

$x_1 = 0 \quad 1 - 2x_1 - 6x_2 = 0$

$2x_2 - 8x_2^2 - 2x_1x_2 = 0$

$x_2(2 - 8x_2 - 2x_1) = 0$

$x_2 = 0 \quad 2 - 8x_2 - 2x_1 = 0$

$\underline{x_1 = 0}$

- $x_2 = 0$
- $2 - 8x_2 - \overset{0}{\cancel{2x_1}} = 0 \Rightarrow x_2 = \frac{1}{4}$

$\underline{x_2 = 0}$

- $x_1 = 0$
- $1 - 2x_1 - \overset{0}{\cancel{6x_2}} = 0 \Rightarrow x_1 = \frac{1}{2}$

$\underline{1 - 2x_1 - 6x_2 = 0} \Rightarrow 2x_1 = 1 - 6x_2$

- $2 - 8x_2 - 2x_1 = 0$

$$2 - 8x_2 - 1 + 6x_2 = 0$$
$$-2x_2 = -1$$
$$x_2 = \frac{1}{2}$$

$$2x_1 = 1 - 6\left(\tfrac{1}{2}\right)$$
$$x_1 = -1$$

Equilibria: $(0,0)$, $(0,\frac{1}{4})$, $(\frac{1}{2},0)$, $(-1,\frac{1}{2})$

2. Compute Jacobian

$$J(x_1, x_2) = \begin{pmatrix} 1 - 4x_1 - 6x_2 & -6x_1 \\ -2x_2 & 2 - 16x_2 - 2x_1 \end{pmatrix}$$

3. Eigenvalues

$J(0,0) = \begin{pmatrix} 1 & 0 \\ \end{pmatrix}$

$J(\frac{1}{2},0) = \begin{pmatrix} 1-2 & -3 \\ 0 & 2-1 \end{pmatrix}$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\lambda_1 = 1, \ \lambda_2 = 2$$

$$J(0,\tfrac{1}{4}) = \begin{pmatrix} 1-\tfrac{6}{4}-\lambda & 0 \\ -\tfrac{1}{2} & 2-4-\lambda \end{pmatrix}$$

$$\lambda_1 = -\tfrac{1}{2}, \ \lambda_2 = -2$$

$$J(\tfrac{1}{2},0) = \begin{pmatrix} 1-2 & -3 \\ 0 & 2-1 \end{pmatrix}$$

$$\lambda_1 = -1, \ \lambda_2 = 1$$

$$J(-1,\tfrac{1}{2}) = \begin{pmatrix} 2 & 6 \\ -1 & -4 \end{pmatrix}$$

$$\det(J(-1,\tfrac{1}{2}) - \lambda I) = (2-\lambda)(-4-\lambda) + 6$$
$$= \lambda^2 + 2\lambda - 2$$

$$\lambda = \frac{-2 \pm \sqrt{4-4(-2)}}{2}$$
$$= -1 \pm \sqrt{3}$$

$$\lambda_1 > 0, \ \lambda_2 < 0$$

4. Classification of Equilibria

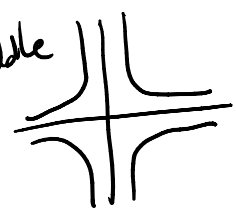
$(0,0)$ $\lambda_1 > 0$, $\lambda_2 > 0$ $\rightarrow$ unstable node

$(0,\tfrac{1}{4})$ $\lambda_1 < 0$, $\lambda_2 < 0$ $\rightarrow$ stable node

$(\tfrac{1}{2},0)$ $\lambda_1 < 0$, $\lambda_2 > 0$ $\longrightarrow$ saddle point

$(-1,\tfrac{1}{2})$ $\lambda_1 > 0$, $\lambda_2 < 0$ $\longrightarrow$ saddle point

Saddle



node