Tangent Planes and Linear Approximations

Recall from 17A

Near \( x_0 \),

\[
f(x) \approx L(x) = f(x_0)(x-x_0) + f(x_0)
\]

Now, in 17C
Near \((x_0, y_0)\),

\[
f(x, y) \approx L(x, y) = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) + f(x_0, y_0)
\]
The Chain Rule

Consider the function $z = f(x, y)$ and the parametric curve $\langle x(t), y(t) \rangle = \vec{s}(t)$.

i.e. The curve in the xy-plane traces out a curve on the surface of $z = f(x, y)$.

Q: How does the height change as you travel around the curve?

i.e. What is the rate of change of your altitude as you travel around the curve?
\[ \frac{dz}{dt} \] But \( z = f(x, y) \) does not directly depend on \( t \), so we need the chain rule!

\[ \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \]

"How your altitude changes..."

\( \frac{\Delta x}{\Delta t} \) \( \frac{\Delta y}{\Delta t} \)
Implicit Differentiation

Recall an implicit function of 2 variables is of the form $F(x,y) = 0$.

Eg. $F(x,y) = x^2 + y^2 - 4$.

What is this? A circle of radius 2!

$F(x,y) = 0$
$x^2 + y^2 = 4$

We can compute $\frac{dy}{dx}$ using the formula

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Now, consider $F(x,y,z) = 0$. This implicitly defines a surface.

Eg. $\partial^2 F(x,y,z) = x^2 + y^2 + z^2 = 9$

What is this? A sphere of radius 3!
We can compute \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) using the formulas

\[
\frac{\partial z}{\partial x} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = - \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}
\]