Solving Systems of Linear Differential Equations

Initial Value Problem (IVP)

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(t_0) = \vec{x}_0$$

General Solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

$\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$ are eigenpairs of A

Steps: 1. Compute eigenvalues of A

2. Compute eigenvectors of A

3. Write down the general solution

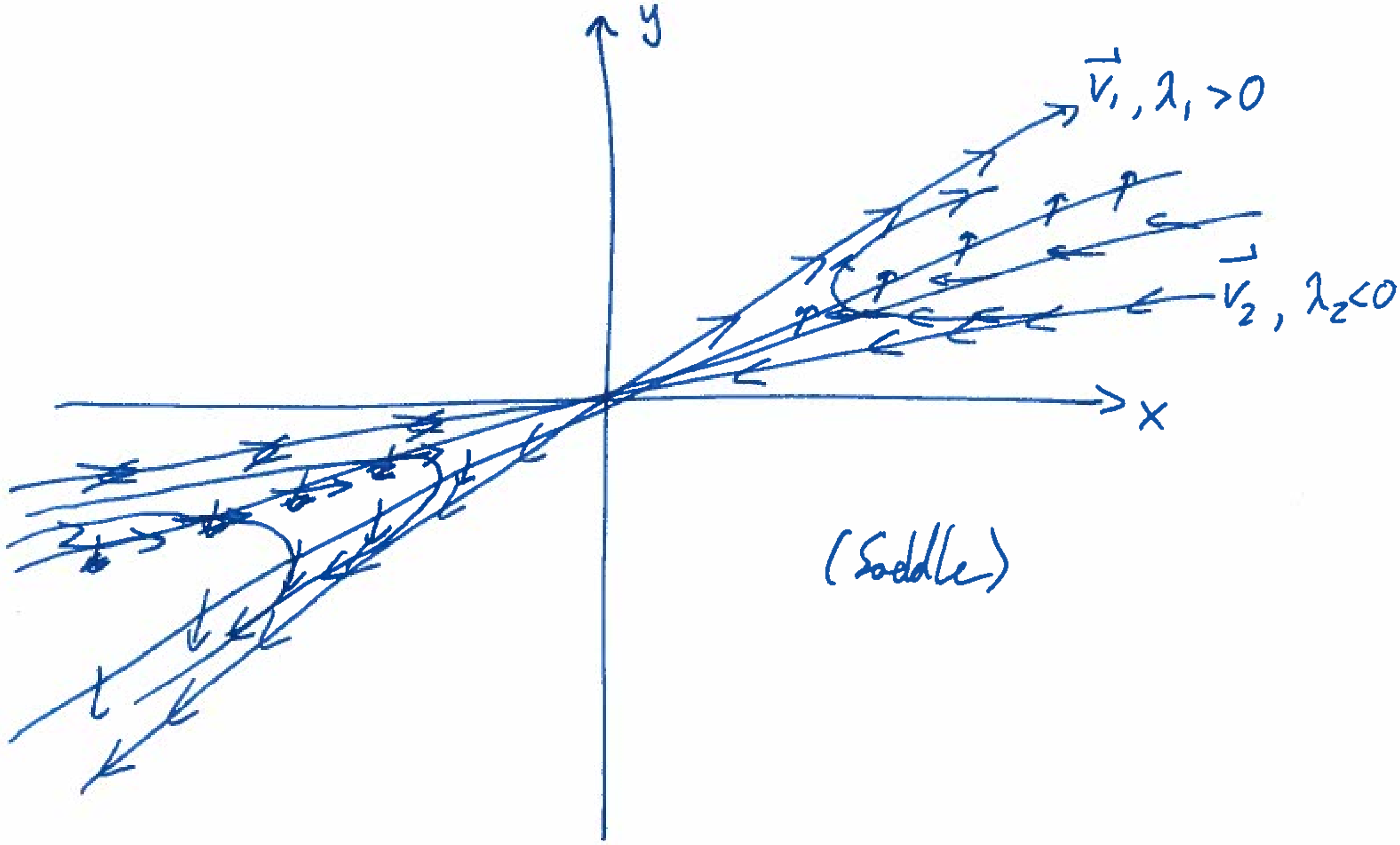
4. Use ICs to determine c_1 and c_2.

What's so special about an eigenvector?

$$A\vec{v} = \lambda\vec{v} \implies \frac{d\vec{x}}{dt} = A\vec{v} = \lambda\vec{v}$$

i.e. If you start on an eigenvector, you stay on an eigenvector!!

Note: A nullcline is where $x' = 0$ or $y' = 0$, so this is where you travel only in the vertical or horizontal direction, respectively.



Long-Term Behavior

What happens as $t \to \infty$??

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

1. $\lambda_1 > 0$ and $\lambda_2 > 0$ (node)
   - ~~decay~~ growth in both directions
2. $\lambda_1 > 0$ and $\lambda_2 < 0$ (saddle)
   - growth in one direction, decay along the other direction
3. $\lambda_1 < 0$ and $\lambda_2 < 0$ (node)
   - decay in both directions.

What happens when $\lambda \in \mathbb{C}$?

i.e. complex eigenvalues

$$e^{\lambda t} = e^{(a+ib)t}$$
$$= e^{at + ibt}$$
$$= e^{at} e^{ibt}$$
$$= e^{at}(\cos(bt) + i\sin(bt))$$

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For general solution, follow same procedure as in real case, but separate terms into real and imaginary parts.

Long-term behavior:

- $a > 0$ (spiral)
  - $|\vec{x}(t)|$ grows
- $a = 0$ (centers/circles)
  - $|\vec{x}(t)|$ is constant

- $a < 0$ (spiral)
    - $|\vec{x}(t)|$ decays