MAT 22A Final Exam (Summer Session I 2020)

1. Determine whether the following statements are true or false. (No need to show work) (20 points, 2 points each)

   (a) Suppose $Q$ is an orthogonal matrix. Then $Q\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = Q^{\mathrm{T}}\mathbf{b}$.

   (b) Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis of some subspace $S$. If $n < k$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is linearly independent, then $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is also a basis of $S$.

   (c) Let $A$ be an $m \times n$ matrix and $n > m$. If $\mathrm{rank}(A) = n$, then $\mathrm{N}(A)$ is $(n - m)$-dimensional.

   (d) The column space of $A$ and the nullspace of $A$ are orthogonal complements.

   (e) If $\lambda = 0$ is an eigenvalue of $A$, then $A$ is invertible.

   (f) Let $A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 4 & 1 & 2 \end{bmatrix}$. $\det(A) = -44$.

   (g) $\det((A^{-1})^{\mathrm{T}}) = \det(A)$

   (h) If $E_{ij}$ is an elimination matrix, then $\det(E_{ij}A) = \det(A)$.

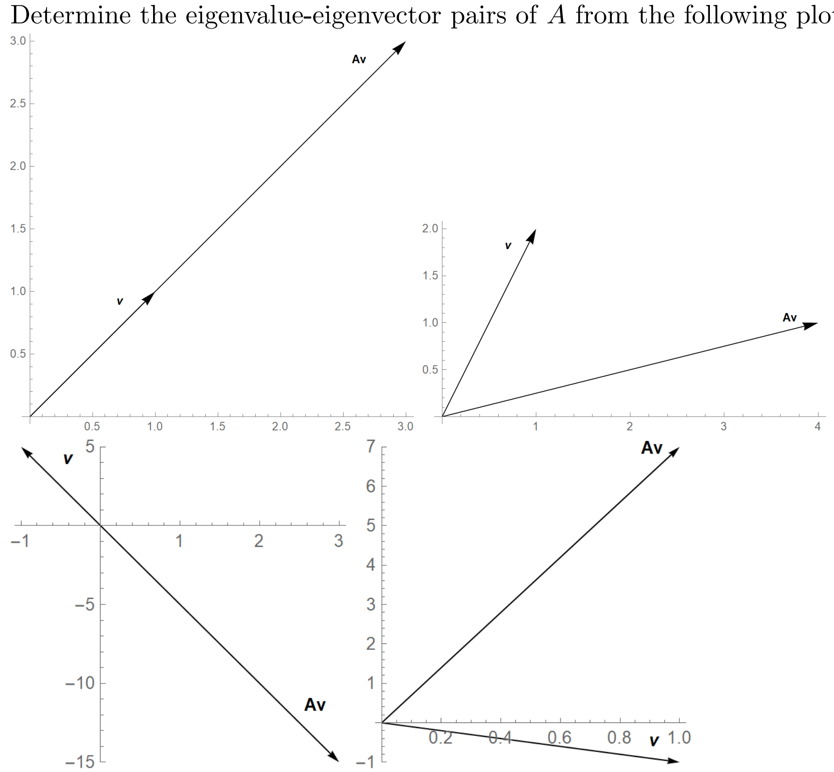
   (i) $\det((AB)^{\mathrm{T}}) = \det(A)\det(B^{\mathrm{T}})$

   (j) If $\lambda$ is an eigenvalue of $A$, then $\frac{1}{\lambda^2}$ is an eigenvalue of $(A^{-1})^2$.

2. Clearly explain your reasoning for the following problems. (25 points)

   (a) Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. (10 points)

   (b) Determine the eigenvalue-eigenvector pairs of $A$ from the following plots. (10 points)




   (c) Compute the determinant of the following matrix. (5 points)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 2 \\ 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 8 & 10 \\ 9 & 2 & 1 & 2 & 3 \end{bmatrix}$$

3. Consider the following matrix $A$. (20 points)

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 4 & -2 \\ 2 & 0 & 0 & 2 \\ 2 & 0 & 2 & 3 \end{bmatrix}$$

(a) Find a basis of the column space of $A$.

(b) Find a basis of the left nullspace of $A$.

(c) Write $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -4 \end{bmatrix}$ as $\mathbf{b} = \mathbf{s} + \mathbf{t}$ where $\mathbf{s} \in \mathrm{C}(A)$ and $\mathbf{t} \in \mathrm{C}(A)^{\perp}$.

(d) What are the dimensions of the four fundamental subspaces of $A$?

4. Find the $A = QR$ decomposition of $A$ where $Q$ is an orthogonal matrix and $R$ is an upper triangular matrix. Then use the $QR$ decomposition to solve $A\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$. (Use back subsitution to solve $R\mathbf{x} = \mathbf{c}$) (20 points)

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

5. Find the $A = X\Lambda X^{-1}$ decomposition of $A^2$. Here $\Lambda$ is a diagonal matrix whose entries are the eigenvalues of $A$ and $X$ is a matrix whose columns are the eigenvectors of $A$. If $B = V \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{-1}$, show that $A$ and $B$ are similar matrices. (10 points)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & -1 \end{bmatrix}$$

6. Find the projection of $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 4 \end{bmatrix}$ onto $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ 7 \\ 1 \end{bmatrix} \right\}$. (10 points)