MAT 22A Final Exam Solutions (Summer Session I 2020)

1. Determine whether the following statements are true or false. (No need to show work) (20 points, 2 points each)

   (a) Suppose Q is an orthogonal matrix. Then $Q\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = Q^{\mathrm{T}}\mathbf{b}$.

   (b) Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis of some subspace S. If $n < k$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is linearly independent, then $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is also a basis of S.

   (c) Let A be an $m \times n$ matrix and $n > m$. If $\mathrm{rank}(A) = n$, then $\mathrm{N}(A)$ is $(n-m)$-dimensional.

   (d) The column space of A and the nullspace of A are orthogonal complements.

   (e) If $\lambda = 0$ is an eigenvalue of A, then A is invertible.

   (f) Let $A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 4 & 1 & 2 \end{bmatrix}$. $\det(A) = -44$.

   (g) $\det((A^{-1})^{\mathrm{T}}) = \det(A)$

   (h) If E_{ij} is an elimination matrix, then $\det(E_{ij}A) = \det(A)$.

   (i) $\det((AB)^{\mathrm{T}}) = \det(A)\det(B^{\mathrm{T}})$

   (j) If λ is an eigenvalue of A, then $\frac{1}{\lambda^2}$ is an eigenvalue of $(A^{-1})^2$.

Solution.

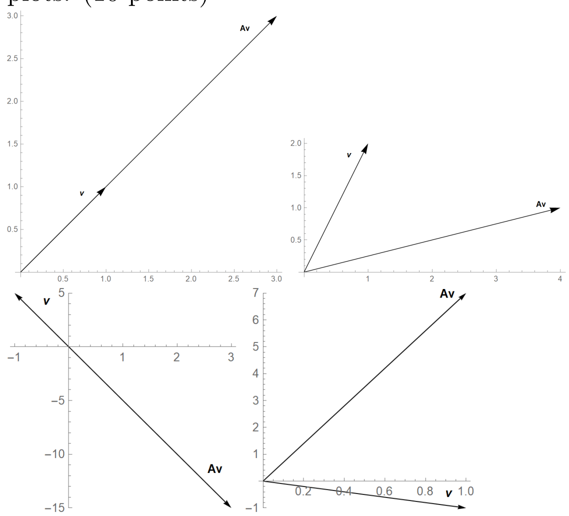
(a) True
(b) False
(c) False
(d) False
(e) False
(f) True
(g) False
(h) True
(i) True
(j) True

□

2. Clearly explain your reasoning for the following problems. (25 points)

(a) Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. (10 points)

(b) Determine the eigenvalue-eigenvector pairs of A from the following plots. (10 points)



(c) Compute the determinant of the following matrix. (5 points)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 2 \\ 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 8 & 10 \\ 9 & 2 & 1 & 2 & 3 \end{bmatrix}$$

Solution.

(a) Consider

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

Multiplying by $\mathbf{v}_1^{\mathrm{T}}$ we get

$$c_1\|v_1\|^2 = \mathbf{0}$$

since $\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 = 0$ by orthogonality. Since $\mathbf{v}_1 \neq \mathbf{0}$, this means that $c_1 = 0$. Then $c_2 = 0$. Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

(b) From the diagram, we see that the top left and bottom left plots show $\mathbf{v}$ and $A\mathbf{v}$ lie on the same line. Therefore, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 3$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ $\lambda = -3$ are the eigenvector eigenvalue pairs of A.

(c)

□

3. Consider the following matrix A. (20 points)

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 4 & -2 \\ 2 & 0 & 0 & 2 \\ 2 & 0 & 2 & 3 \end{bmatrix}$$

(a) Find a basis of the column space of A.

(b) Find a basis of the left nullspace of A.

(c) Write $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -4 \end{bmatrix}$ as $\mathbf{b} = \mathbf{s} + \mathbf{t}$ where $\mathbf{s} \in \mathrm{C}(A)$ and $\mathbf{t} \in \mathrm{C}(A)^{\perp}$.

(d) What are the dimensions of the four fundamental subspaces of A?

Solution.

(a) We find that

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \end{bmatrix}$ form a basis of $\mathrm{C}(A)$.

(b) We solve $\mathrm{rref}(A^{\mathrm{T}})\mathbf{x} = \mathbf{0}$ and find

$$\mathrm{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(c)

(d) The column space and row space are 3-dimensional. Then the left nullspace and the nullspace are 1 dimensional.

□

4. Find the $A = QR$ decomposition of A where Q is an orthogonal matrix and R is an upper triangular matrix. Then use the QR decomposition to solve $A\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$. (Use back subsitution to solve $R\mathbf{x} = \mathbf{c}$) (20 points)

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

Solution.

First, to find Q, we use Gram-Schmidt to find an orthonormal basis of C(A). We see that the solutions of A are linearly independent, so first we orthogonalize.

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{w}_2 &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}}{\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \|^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}. \end{aligned}$$

Now, normalizing the vectors, we find that

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

so

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

Now,

$$\begin{aligned} R &= Q^{\mathrm{T}} A \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix}. \end{aligned}$$

Now,

$$R\mathbf{x} = Q^{\mathrm{T}} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

so

$$\begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Thus,

$$\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

□

5. Find the $A = X\Lambda X^{-1}$ decomposition of A^2. Here Λ is a diagonal matrix whose entries are the eigenvalues of A and X is a matrix whose columns are the eigenvectors of A. If $B = V \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{-1}$, show that A and B are similar matrices. (10 points)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & -1 \end{bmatrix}$$

Solution.

First, we find the eigenvalues of A.

$$\det(A - \lambda I) = (-1 - \lambda)(1 - \lambda)(-\lambda)$$

so $\lambda_1 = -1$, $\lambda_2 = 1$, and $\lambda_3 = 0$. Now we find the eigenvalues of A. for $\lambda_1 = -1$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \mathbf{v}_1 = \mathbf{0}$$

so $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Next, $\lambda_2 = 1$ we have

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 2 & -1 & -2 \end{bmatrix} \mathbf{v}_2 = \mathbf{0}$$

so $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Finally $\lambda_3 = 0$, we have

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & -1 \end{bmatrix} \mathbf{v}_3 = \mathbf{0}$$

so $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. □

Thus,

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and we find that

$$X^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $A = X\Lambda X^{-1}$, so $A^2 = X\Lambda^2 X^{-1}$. Now, since $B = V\Lambda V^{-1}$ and $A = X\Lambda X^{-1}$, we see that $B = VX^{-1}AXV^{-1}$. Let $C = VX^{-1}$. Then $C^{-1} = XV^{-1}$. Thus, $B = CBC^{-1}$, so A and B are similar matrices.

6. Find the projection of $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 4 \end{bmatrix}$ onto $S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ 7 \\ 1 \end{bmatrix} \right\}$.

(10 points)

Solution.

First, we need to determine whether the given vectors are linearly independent. We have

$$\begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 7 \\ 2 & 1 & 7 \\ 1 & -2 & 1 \end{bmatrix} \begin{matrix} \\ r_2 - 2r_1 \\ r_3 - 2r_1 \\ r_4 - r_1 \end{matrix} = \begin{bmatrix} 1 & -2 & -4 \\ 0 & 5 & 15 \\ 0 & 5 & 15 \\ 0 & 0 & 5 \end{bmatrix} \begin{matrix} r_1 + \frac{2}{5} r_2 \\ \\ r_3 - r_2 \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\vdots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

First,

$$A^{\mathrm{T}} A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ -2 & 1 & 1 & -2 \\ -4 & 7 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 7 \\ 2 & 1 & 7 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 23 \\ 0 & 10 & 20 \\ 25 & 20 & 115 \end{bmatrix}$$

and so

$$(A^{\mathrm{T}} A)^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{21}{50} & -\frac{4}{25} \\ -\frac{1}{5} & -\frac{4}{25} & \frac{2}{25} \end{bmatrix}.$$

Now,

$$\mathbf{p} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 7 \\ 2 & 1 & 7 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{21}{50} & -\frac{4}{25} \\ -\frac{1}{5} & -\frac{4}{25} & \frac{2}{25} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ -2 & 1 & 1 & -2 \\ -4 & 7 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

□