MAT 22A Midterm Solutions (Summer Session I 2020)

1. Determine whether the following statements are true or false. (No work required) (18 points)

   (a) The linear combination $a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ fills a line. a and b are real numbers.

   (b) The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ are orthogonal.

   (c) $\mathbf{x} = \mathbf{0}$ is a solution to $A\mathbf{x} = \mathbf{0}$.

   (d) If A is an $k \times n$ matrix and B is an $n \times \ell$ matrix, then $(AB)^{\mathrm{T}}$ is an $\ell \times k$ matrix.

   (e) If A and B are symmetric matrices, then $AB = BA$.

   (f) If the columns of A are linearly dependent, then A is invertible.

   (g) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} — & \mathbf{a}_1 & — \\ — & \mathbf{a}_2 & — \\ — & \mathbf{a}_3 & — \end{bmatrix}$ produces the matrix $\begin{bmatrix} — & \mathbf{a}_1 & — \\ — & \mathbf{a}_3 & — \\ — & \mathbf{a}_2 & — \end{bmatrix}$.

   (h) The set of $n \times n$ upper triangular matrices are a subspace of the vector space of $n \times n$ matrices.

   (i) If A is an $m \times n$ matrix and $m < n$, then $A\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Solution.

(a) **False**, the linear combination fills a plane.

(b) **False**, the dot product of the two vectors is not zero.

(c) **True**, $\mathbf{x} = \mathbf{0}$ is the trivial solution to $A\mathbf{x} = \mathbf{0}$.

(d) **True**, AB produces a $k \times \ell$ matrix, so $(AB)^{\mathrm{T}}$ is $\ell \times k$.

(e) **False**, if AB is symmetric, then the statement holds.

(f) **False**, A is invertible if and only if the columns of A and linearly independent.

(g) **True**, the given matrix is a permutation matrix

(h) **True**, we showed that this is true for lower triangular matrices.

(i) **True**, A will always have a free column if $m < n$.

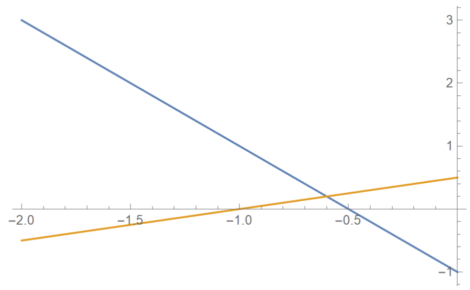
□

2. Sketch the column picture and row picture of the following system. (7 points)

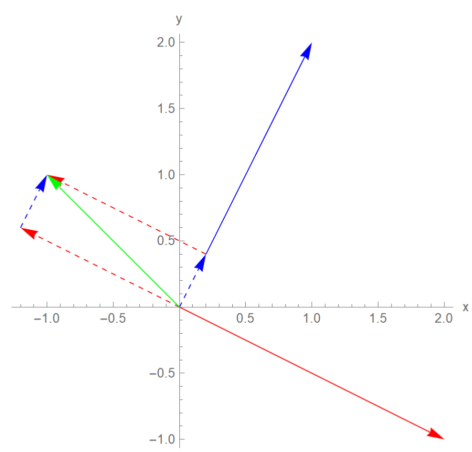
$$2x + y = -1$$
$$-x + 2y = 1$$

Solution.

The column picture is the intersection of the lines $2x + y = -1$ and $-x + 2y = 1$.



The row picture is using the columns of the coefficient matrix to produce the vector on the right-hand side. That is, we want to use the red and blue vector to produce the green vector.



□

3. Solve the following system using the inverse of the coefficient matrix. (10 points)

$$\begin{aligned} 2x_1 - 4x_2 - 6x_3 &= 0 \\ x_1 + 2x_2 + 1x_3 &= 4 \\ -2x_1 + 4x_3 &= -1 \end{aligned}$$

Solution.

We find the inverse of the coefficient matrix using Gauss-Jordan elimination and the aumented matrix

$$\left[\begin{array}{ccc|ccc} 2 & -4 & -6 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -2 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ r_2 - \frac{1}{2}r_1 \\ r_3 + r_1 \end{array} = \left[\begin{array}{ccc|ccc} 2 & -4 & -6 & 1 & 0 & 0 \\ 0 & 4 & 4 & -\frac{1}{2} & 1 & 0 \\ 0 & -4 & -2 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} r_1 + r_2 \\ \\ r_3 + r_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & -2 & \frac{1}{2} & 1 & 0 \\ 0 & 4 & 4 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 2 & \frac{1}{2} & 1 & 1 \end{array}\right] \begin{array}{c} r_1 + r_3 \\ r_2 - 2r_3 \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 2 & 1 \\ 0 & 4 & 0 & -\frac{3}{2} & -1 & -2 \\ 0 & 0 & 2 & \frac{1}{2} & 1 & 1 \end{array}\right] \begin{array}{c} \frac{1}{2}r_1 \\ \frac{1}{4}r_2 \\ \frac{1}{2}r_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{3}{8} & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Now, the solution to the system is

$$\mathbf{x} = A^{-1} \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}.$$

□

4. Find the LU-decomposition of the coefficient matrix and use back-substitution followed by forward-substitution to solve the following system. (15 points)

$$\begin{aligned} 4x_1 + 2x_2 + -2x_3 &= 4 \\ 2x_1 - x_2 - x_3 &= 6 \\ 3x_1 + x_3 &= -7 \end{aligned}$$

Solution.

First, we find the LU-decomposition of the coefficient matrix. We begin by finding the upper triangular matrix U.

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & -1 & -1 \\ 3 & 0 & 1 \end{bmatrix} \begin{matrix} \\ r_2 - \frac{1}{2}r_1 \\ r_3 - \frac{3}{4}r_1 \end{matrix} = \begin{bmatrix} 4 & 2 & -2 \\ 0 & -2 & 0 \\ 0 & -\frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{matrix} \\ \\ r_3 - \frac{3}{4}r_2 \end{matrix}$$
$$= \begin{bmatrix} 4 & 2 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}.$$

We used the multipliers $\ell_{21} = \frac{1}{2}$, $\ell_{31} = \frac{3}{4}$, and $\ell_{32} = \frac{3}{4}$, so the lower triangular matrix L is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{4} & \frac{3}{4} & 1 \end{bmatrix}.$$

Thus, the LU decomposition of the coefficient matrix is

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & -1 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{4} & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}.$$

Now, let $U\mathbf{x} = \mathbf{c}$. First, we use forward-substitution to solve $L\mathbf{c} = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$.

We have

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{4} & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$$

so

$$\begin{aligned} c_1 &= 4 \\ c_2 &= 6 - \frac{1}{2}c_1 = 4 \\ c_3 &= -7 - \frac{3}{4}c_1 - \frac{3}{4}c_2 = -13. \end{aligned}$$

Now, we use back-substitution to solve $U\mathbf{x} = \mathbf{c}$. We have

$$\begin{bmatrix} 4 & 2 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -13 \end{bmatrix}$$

so

$$\begin{aligned} x_3 &= -\frac{26}{5} \\ x_2 &= -2 \\ x_1 &= \frac{1}{4}(4 - 2x_2 + 2x_3) = -\frac{3}{5}. \end{aligned}$$

Thus $\mathbf{x} = \begin{bmatrix} -\frac{3}{5} \\ -2 \\ -\frac{26}{5} \end{bmatrix}$. □

5. Find the nullspace of the following matrix and determine the rank of A. (10 points)

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \end{bmatrix}$$

Solution.

We want to find solutions to $A\mathbf{x} = \mathbf{0}$. First, we find rref(A).

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \end{bmatrix} \begin{matrix} \\ r_2 - 2r_1 \\ r_3 - r_1 \end{matrix} &= \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -5 & -2 & -1 \\ 0 & -2 & -2 & 1 & -2 \end{bmatrix} \begin{matrix} \\ \\ r_3 + 2r_2 \end{matrix} \\
&= \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -5 & -2 & -1 \\ 0 & 0 & -12 & -3 & -4 \end{bmatrix} \begin{matrix} r_1 + \frac{1}{6}r_3 \\ r_2 - \frac{5}{12}r_3 \\ \\ \end{matrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{2}{3} \\ 0 & 0 & -12 & -3 & -4 \end{bmatrix} \begin{matrix} \\ \\ -\frac{1}{12}r_3 \end{matrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{3} \end{bmatrix}.
\end{aligned}$$

We see that

$$\begin{aligned}
x_1 &= -\frac{1}{2}x_4 - \frac{1}{3}x_5 \\
x_2 &= \frac{3}{4}x_5 - \frac{2}{3}x_5 \\
x_3 &= -\frac{1}{4}x_5 - \frac{1}{3}x_5
\end{aligned}$$

so

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_4 - \frac{1}{3}x_5 \\ \frac{3}{4}x_5 - \frac{2}{3}x_5 \\ -\frac{1}{4}x_5 - \frac{1}{3}x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}.$$

Thus the nullspace of A is the span of $\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$. From the rref($A$), we see that the rank of A is 3.

□