Ch. 1 Introduction to Vectors

1.1 Vectors and Linear Combinations

__Definition__ A vector $\vec{u} \in \mathbb{R}^n$ is an ordered n-tuple with entries $u_i \in \mathbb{R}$. The u_i are scalars.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{(Column vector)}$$

__Notation__ 1. Vectors are denoted by boldface or an arrow

2. $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ is a column vector, but we will also write

$$\vec{u} = (u_1, u_2, \ldots, u_n)$$

to denote $\vec{u}$.

3. This is different from

$$\vec{w} = [w_1, w_2, \ldots, w_n]$$

which is a row vector.

Manipulating Vectors Let $\vec{v}, \vec{w} \in \mathbb{R}^n$

1. Vector addition

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

2. Scalar Multiplication $(c \in \mathbb{R})$

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix} = c \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The vector $\vec{0}$ is called the zero vector and all of its entries are zero.

$$\vec{0} = (0, 0, \ldots, 0)$$

<u>Ex</u> Let $\vec{v} \in \mathbb{R}^3$; $v_i = i$ for $i = 1, 2, 3$. Let $\vec{w} \in \mathbb{R}^3$ and $w_i = i+1$. Compute $\vec{u} = \vec{v} + \vec{w}$.

$$\begin{aligned} u_i &= v_i + w_i \\ &= i + i + 1 \\ &= 2i + 1 \end{aligned}$$

$$\vec{u} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Alternatively,

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

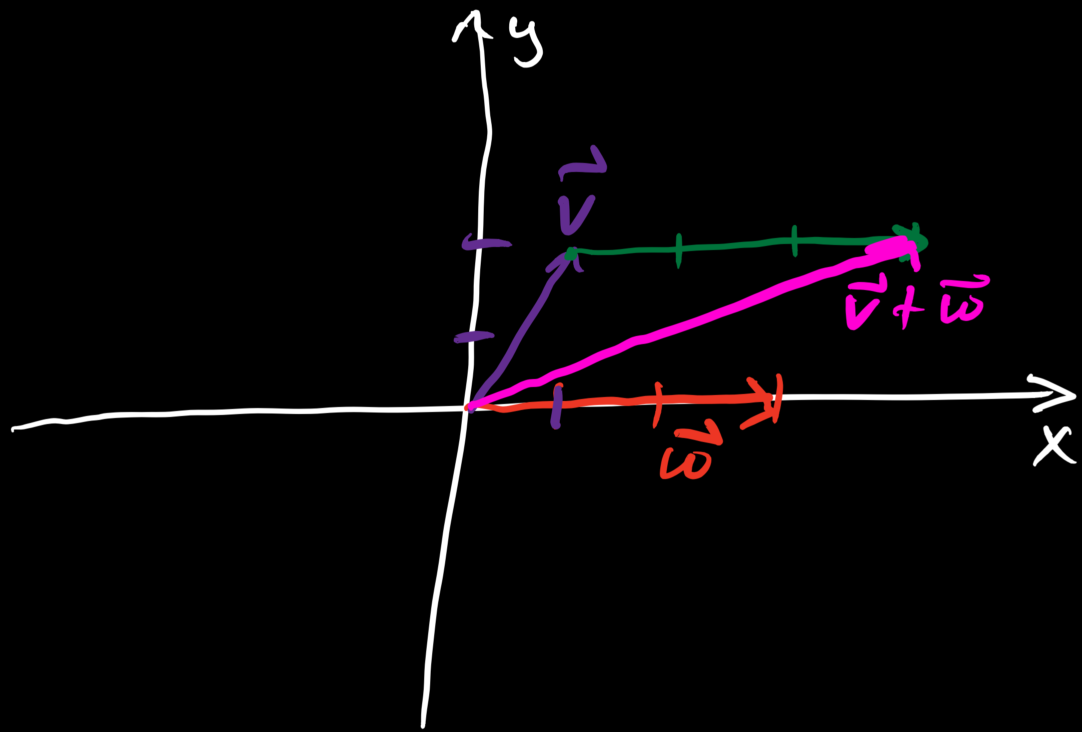
so

$$\begin{aligned} \vec{u} &= \vec{v} + \vec{w} \\ &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \end{aligned}$$

Ways to represent a vector $\vec{v}$

- by its entries
- arrow from the origin
- point in the plane

Ex Let $\vec{w} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.



- for vector addition of $\vec{v}$ and $\vec{w}$, place the tail of $\vec{w}$ at the head of $\vec{v}$

Linear Combination

Defn Let $a, b \in \mathbb{R}$ and $\vec{u}, \vec{v} \in \mathbb{R}^n$.
The sum of $a\vec{u}$ and $b\vec{v}$

is called a linear combination.

$$a\vec{u} + b\vec{v}$$

<u>Ex</u>

$$\underset{?}{2}\underset{\vec{u}}{\begin{bmatrix}1\\2\\1\end{bmatrix}} + \underset{?}{3}\underset{\vec{v}}{\begin{bmatrix}1\\3\\0\end{bmatrix}} = \underset{\vec{w}}{\begin{bmatrix}5\\13\\2\end{bmatrix}}$$

$\vec{w}$ is formed by a linear combination of $\vec{u}$ and $\vec{v}$

<u>Question</u>

In 3-D, consider the linear combinations

$$a\vec{u} + b\vec{v} + c\vec{w}$$

where $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and a, b, c are any values in $\mathbb{R}$. What does the linear combination "fill"?

<u>Answer</u>: It depends on $\vec{u}$, $\vec{v}$, and $\vec{w}$!

1. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are the zero vector, we only get the zero vector.

2. Say $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent (to be defined later), then

   a) $a\vec{u}$ fills a line through the origin

   b) $a\vec{u} + b\vec{v}$ fills a plane through the origin

c) $a\vec{u} + b\vec{v} + c\vec{w}$ fill three dimensional space

Question How do we determine if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are "linearly independent"?

Answer First lets only consider $\vec{u}$ and $\vec{v}$. If $a\vec{u} + b\vec{v}$ do not fill a line, then they fill a plane.

• $\vec{u}, \vec{v}$ nonzero

Now if $\vec{w}$ does not lie in the plane filled by

$$a\vec{u} + b\vec{v},$$

then

$$a\vec{u} + b\vec{v} + c\vec{w}$$

fills 3-space.

1.2 Lengths and Dot Products

Defn Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. The dot product $\vec{v} \cdot \vec{w}$ is

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i$$
$$= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Remarks 1. We can think of the dot product as a way to multiply vectors

2. The dot product produces a scalar, NOT a vector.

Ex Let $\vec{u} = (1, 2, 0)$ and $\vec{v} = (-2, 1, 0)$.
Then

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 1 \cdot (-2) + 2(1) + 0 \cdot 0 \\ &= -2 + 2 \\ &= 0 \end{aligned}$$

Note If $\vec{u} \cdot \vec{v} = 0$, then we say that $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) and write $\vec{u} \perp \vec{v}$.

Lengths and Unit Vectors

Defn The norm (or length) of a vector is

$$\begin{aligned} \|\vec{v}\| &= \sqrt{\vec{v} \cdot \vec{v}} \\ &= \sqrt{\sum_{i=1}^{n} v_i^2} \end{aligned}$$

Defn A unit vector $\vec{u}$ is a vector whose length is one.

$$\|\vec{u}\| = 1$$

Question: How do we find a unit vector that points in the same direction as some vector $\vec{v}$?

Answer: Normalize $\vec{v}$! (i.e. multiply $\vec{v}$ by $\frac{1}{\|\vec{v}\|}$)

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

$$\|\vec{u}\| = \left\|\frac{1}{\|\vec{v}\|}\vec{v}\right\|$$

$$= \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1$$

<u>Ex</u> Find a unit vector $\vec{u}$ that points in the same direction as

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

First lets find $\|\vec{v}\|$.

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

$$= \sqrt{1 + 2^2 + 3^2 + 4^2}$$

$$= \sqrt{30}$$

Then

$$\vec{u} = \frac{1}{\sqrt{30}} \vec{v}$$

$$= \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Proposition Suppose $\vec{u}$ and $\vec{v}$ are orthogonal. Then

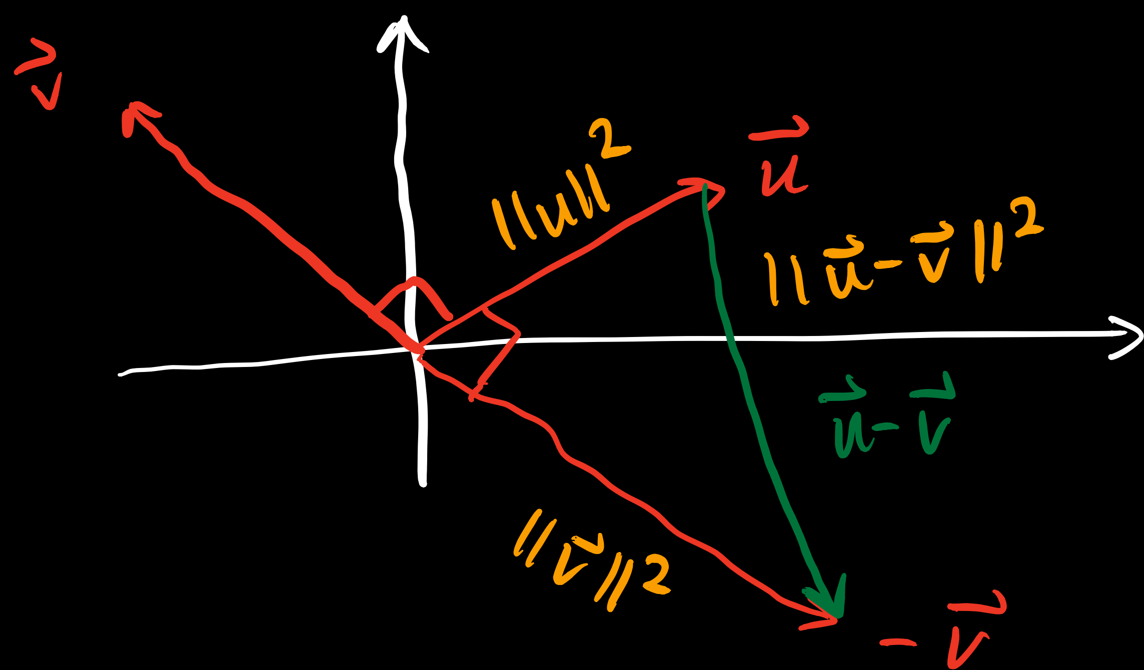
$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} - \vec{v}\|^2.$$

Proof We will show RHS = LHS

$$
\begin{aligned}
\|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\
&= \sum_{i=1}^{n} (u_i - v_i)^2 \\
&= \sum_{i=1}^{n} u_i^2 - 2u_i v_i + v_i^2 \\
&= \sum_{i=1}^{n} u_i^2 - \sum_{i=1}^{n} 2u_i v_i + \sum_{i=1}^{n} v_i^2 \\
&= \vec{u} \cdot \vec{u} - 2 \sum_{i=1}^{n} u_i v_i + \vec{v} \cdot \vec{v} \\
&= \|\vec{u}\|^2 - 2\, \vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 - 2(0) + \|\vec{v}\|^2
\end{aligned}
$$

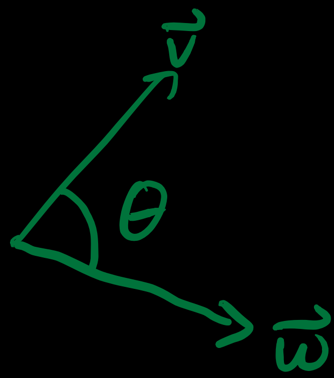
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2$$

<u>Remark</u> The above proposition is the familiar Pythagorean theorem.



The dot product can also tell us the amount of overlap between two vectors using the cosine formula

$$\frac{\vec{v}}{\|\vec{v}\|} \cdot \frac{\vec{w}}{\|\vec{w}\|} = \cos\theta$$



Two famous inequalities

Schwarz inequality:

$$|\vec{v}\cdot\vec{w}| \le \|\vec{v}\|\,\|\vec{w}\|$$

Triangle Inequality:

$$\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$$

<u>Ex</u> Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Find the angle between $\vec{v}$ and

$\vec{w}$ and check the inequalities.

$$\cos\theta = \frac{1}{\|\vec{v}\|}\begin{bmatrix}1\\1\end{bmatrix} \cdot \frac{1}{\|\vec{w}\|}\begin{bmatrix}-2\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{5}}(-2+1)$$

$$= \frac{-1}{\sqrt{10}}$$

$$|\vec{w}\cdot\vec{v}| = |-2+1|$$

$$= 1$$

$$\|\vec{v}\| = \sqrt{2}, \quad \|\vec{w}\| = \sqrt{5}$$

$$|\vec{w}\cdot\vec{v}| = 1 \leq \sqrt{2}\sqrt{5} = \|\vec{w}\|\|\vec{v}\|$$

$$\vec{v} + \vec{w} = \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\|\vec{v} + \vec{w}\| = \sqrt{1+4}$$

$$= \sqrt{5}$$

$$\|\vec{v} + \vec{w}\| = \sqrt{5} \leq \sqrt{2} + \sqrt{5} = \|\vec{v}\| + \|\vec{w}\|$$

indeed

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$$