Chapter 4 Orthogonality

4.1 Orthogonality of the Four Subspaces

We introduced the four fundamental subspaces (row, column, null, left null-space). We will see that the property of orthogonality will give us "nice" bases and matrices.

Recall 1. $\vec{v}$ and $\vec{w}$ are orthogonal if

$$\vec{v}^T\vec{w} = \vec{v}\cdot\vec{w} = 0.$$

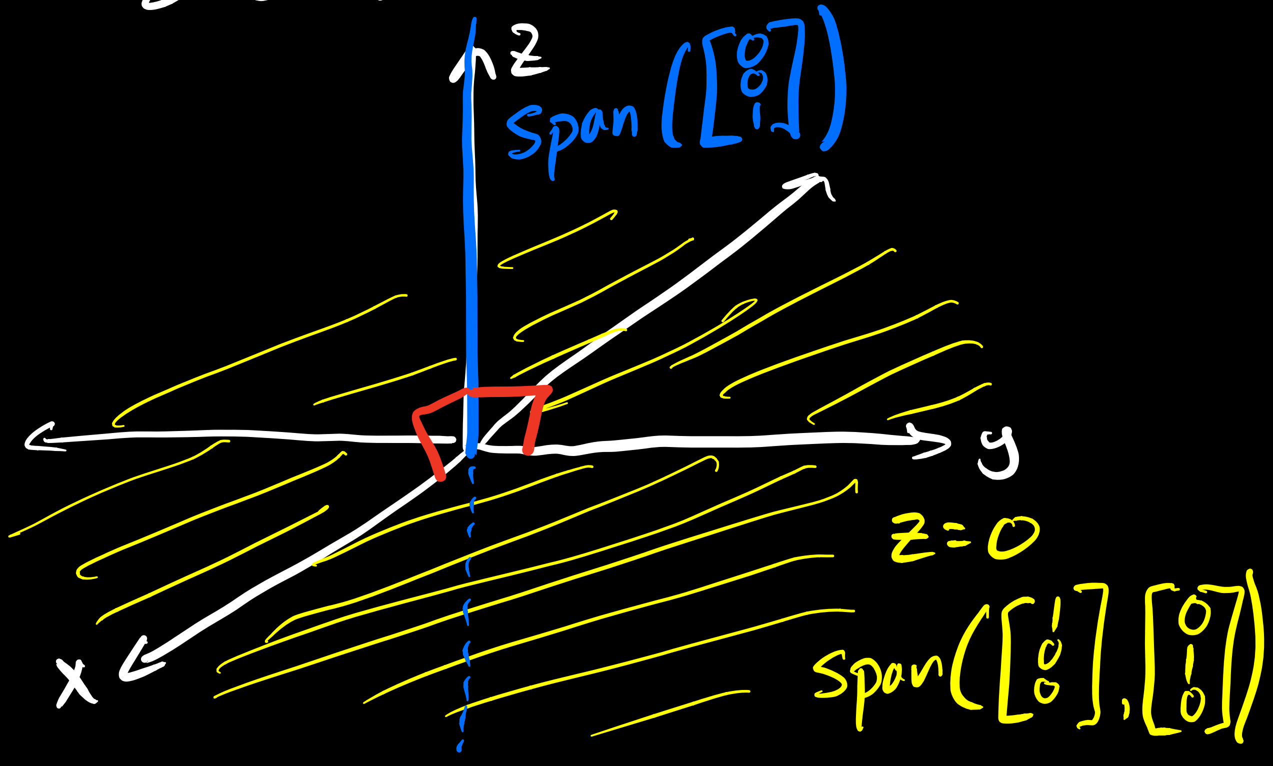
2. Pythagorean theorem

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v}+\vec{w}\|^2$$

3. If a vector $\vec{v}$ is orthogonal to itself then $\vec{v}$ is the zero vector.

Definition Two subspaces V and W of a vector space are orthogonal if every vector $\vec{v} \in V$ is perpendicular to every $\vec{w} \in W$.

Ex Consider $\mathbb{R}^3$ and the subspaces $z=0$ and line $\vec{x}(t) = (0,0,t)$.

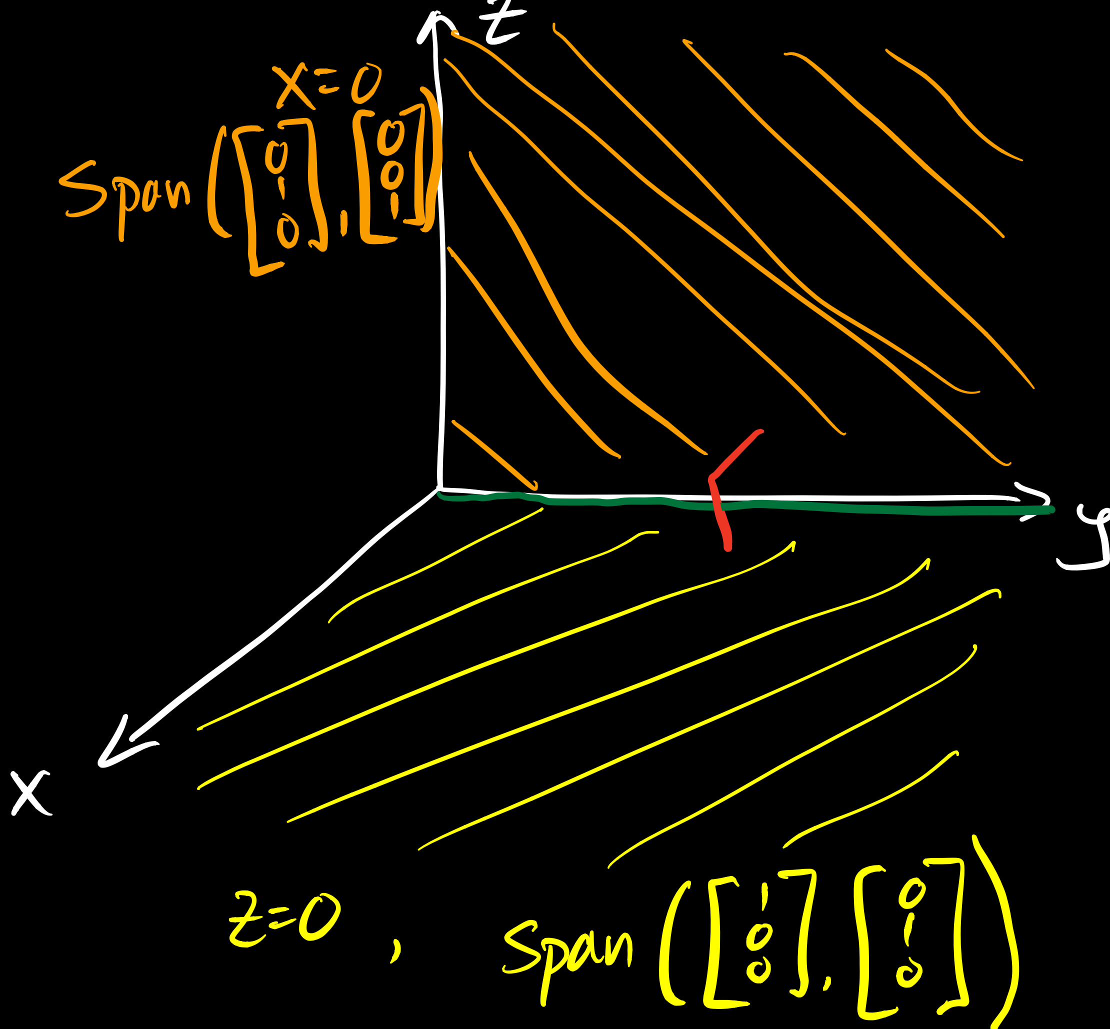


Vectors in the $z=0$ plane look like $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ and vectors on the line look like $\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$. Now,

$$\begin{bmatrix} a & b & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = 0$$

So the subspaces are orthogonal.

<u>Note</u> Consider $z=0$ and $x=0$ as subspaces of $\mathbb{R}^3$. Are these subspaces orthogonal?



$\begin{bmatrix}0\\1\\0\end{bmatrix}$ is in both subspaces,

and $\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \begin{bmatrix}0\\1\\0\end{bmatrix} = 1$,

So the subspaces are not orthogonal.

<u>Proposition</u> The nullspace of A and rowspace of A are orthogonal subspaces of $\mathbb{R}^n$.

<u>Proof</u> Let A be an $m \times n$ matrix and

$$A = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix}.$$ If $\vec{x} \in N(A)$, then

$A\vec{x} = \vec{0}$.

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix} \vec{x}$$

$$= \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix}$$

$= \vec{0}$

So $\vec{x}$ is orthogonal to the rows of A. Since $\vec{x}$ is arbitrary, we have shown that any vector in $N(A)$ is orthogonal to the rows of A. Thus $N(A)$ and $C(A^T)$ are orthogonal subspaces.

Remark The left nullspace $N(A^T)$ and the column space $C(A)$ are orthogonal in $\mathbb{R}^m$.

Ex Find a basis of the row space and nullspace. Show that $C(A^T)$ and $N(A)$ are orthogonal.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix} r_2 - 2r_1$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -3 \end{bmatrix} r_1 + \frac{4}{3} r_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} -\frac{1}{3} r_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

First find $N(A)$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$x_1 = -2x_2$$

$$x_3 = 0$$

$$\vec{x} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

so $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ forms a basis of $N(A)$.

$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis of $C(A^T)$.

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \left(c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= c_1 c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_1 c_3 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= 0$$

Orthogonal Complements

We have seen that $N(A)$ and $C(A^T)$ are orthogonal subspaces. There is a deeper connection that we may subspeet from the rank-nullity theorm.

<u>Definition</u> The orthogonal complement of a subspace V contains every vector that is perpendicular to V, denoted $V^\perp$.

<u>Ex 1.</u> $N(A)$ is the collection of $\vec{x}$ so that $\vec{x}$ is orthogonal to the rows of A.

Then $C(A^T)^\perp = N(A)$.

<u>Ex 2</u> What is $N(A)^\perp$? $C(A^T)$!

If $\vec{v} \in N(A)^\perp$, then it must be in $C(A^T)$. If not, then we can add $\vec{v}$ as an additional row of A without changing the nullspace, but by the rank-nullity theorem, we must have $r+(n-r)=n$.

<u>Remark</u> 1. The left nullspace and column space are orthogonal complements.

2. The only vector in two orthogonal subspaces is the zero vector.

Pseudoinverse of a matrix

A rank r matrix has an $r\times r$ invertible submatrix. The pseudoinverse will invert the submatrix.

Ex

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

$$\rightsquigarrow$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The submatrix is $\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$

and

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{5-8} \begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5/3 & 4/3 \\ 2/3 & -1/3 \end{bmatrix}$$

$$\begin{bmatrix} -5/3 & 4/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{3}+\frac{8}{3} & -\frac{10}{3}+\frac{16}{3} & -\frac{20}{3}+\frac{20}{3} \\ \frac{2}{3}-\frac{2}{3} & \frac{4}{3}-\frac{4}{3} & \frac{8}{3}-\frac{5}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \mathrm{rref}(A).$$

Combining Bases from Subspaces

Facts about bases: 1. Any n linearly independent vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$, so they are a basis.

2. Any n vectors that span $\mathbb{R}^n$ must be independent, so they are a basis.

Then

- If n columns of A are independent, they must span $\mathbb{R}^n$, so

$$A\vec{x} = \vec{b} \text{ has a solution.}$$

- If the n columns span $\mathbb{R}^n$, they are independent. So $A\vec{x} = \vec{b}$ has only one solution.

Ex

$$A = \begin{bmatrix} 2 & 10 \\ 5 & 25 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 10 & 2 \\ 5 & 25 & 5 \end{array}\right] r_2 - \frac{5}{2} r_1$$

$$\left[\begin{array}{cc|c} 2 & 10 & 2 \\ 0 & 0 & 0 \end{array}\right] \tfrac{1}{2} r_1$$

$$\left[\begin{array}{cc|c} 1 & 5 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 = 1 - 5x_2$$

$$\vec{x} = \begin{bmatrix} 1 - 5x_2 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

$$\vec{x} = x_p + x_n$$