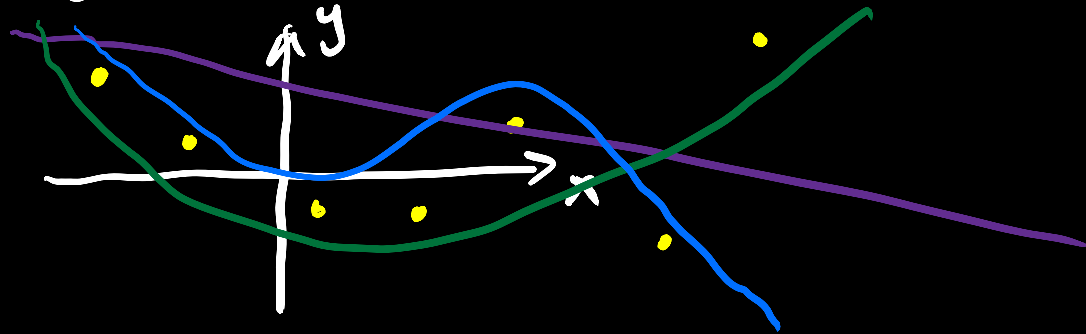
Application: Polynomial Interpolation
(Best fit lines, parabolas, cubics, ...)

Suppose we have some collection of data $(t_1, b_1), (t_2, b_2), \ldots, (t_m, b_m)$. We want to fit the data using the n-degree polynomial $\sum_{i=0}^{n} c_i x^i$. Then we want

$$c_0 + c_1 t_1 + c_2 t_1^2 + \cdots + c_n t_1^n = b_1$$

$$\vdots$$

$$c_0 + c_1 t_m + c_2 t_m^2 + \cdots + c_n t_m^n = b_m$$



Then we have the system

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^n \\ 1 & t_2 & t_2^2 & \cdots & t_2^n \\ \vdots & & & & \\ 1 & t_m & t_m^2 & \cdots & t_m^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

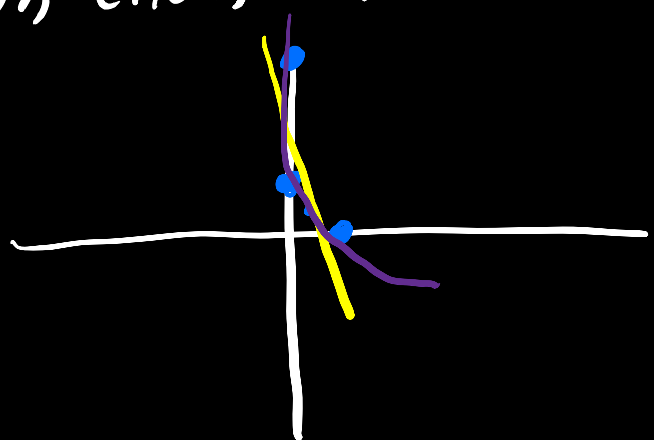
$$T\vec{c} = \vec{b}$$

There may be no solution $\vec{c}$. We can find a curve which minimizes the distance between the curve and points by finding the least squares solution $\hat{\vec{c}}$ by solving

$$T^T T \hat{\vec{c}} = T^T \vec{b}$$

Error : $\| \vec{b} - T\vec{c} \|$

<u>Ex</u> Find the best-fit curve through $(0,6)$, $(1,0)$, $(0,2)$ and $(\frac{1}{2}, 1)$.



First let's find a best fit line.

$$c_0 + c_1 t$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1/2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 42/11 \\ -46/11 \end{bmatrix}$$

Best fit line: $\frac{42}{11} - \frac{46}{11} t$

What about the best fit parabola?

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

4.4 Orthonormal Bases and Gram-Schmidt

We have seen how bases give us a way to describe vectors in a subspace and how orthogonality was a key idea in finding projections. Now, we look at orthonormal bases which will give us nice projection matrices.

<u>Definition</u> The vectors $\vec{q}_1, \ldots, \vec{q}_n$ are orthonormal if

$$\vec{q}_i^{\,T}\vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \text{ (orthog.)} \\ 1 & \text{if } i = j \text{ (unit vector)} \end{cases}$$

Let Q be a matrix with orthonormal columns. $Q = \begin{bmatrix} | & & | \\ \vec{q}_1 & \cdots & \vec{q}_n \\ | & & | \end{bmatrix}$. When

Q is a square matrix, we call Q an orthogonal matrix.

<u>Properties of Q:</u>

1. $Q^TQ = I$

$$(Q^TQ)_{ij} = \vec{q}_i^{\,T}\vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then $$Q^TQ = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I$$

2. If Q is square, $Q^T = Q^{-1}$

We know $Q^TQ = I$. For square

matrices, $QQ^T = I$. Then

$Q^TQ = QQ^T = I$, so $Q^T = Q^{-1}$!

3. Q preserves lengths and angles

a) $\|Q\vec{x}\| = \|\vec{x}\|$

<u>Proof</u> $\|Q\vec{x}\|^2 = (Q\vec{x})^T(Q\vec{x})$

$$= \vec{x}^T Q^T Q \vec{x}$$

$$= \vec{x}^T \vec{x}$$

$$= \|\vec{x}\|^2$$

b) $(Q\vec{x}) \cdot (Q\vec{y}) = \vec{x} \cdot \vec{y}$

<u>Proof</u> $(Q\vec{x})^T(Q\vec{y}) = \vec{x}^T Q^T Q \vec{y}$

$$= \vec{x}^T \vec{y}$$

Ex 1 Rotations

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

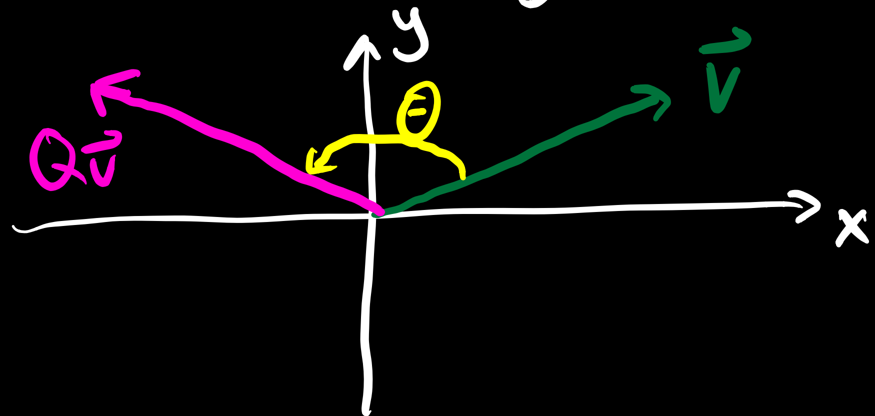
$$\begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = -\sin\theta\cos\theta + \sin\theta\cos\theta$$

$$= 0$$

$$\|(\cos\theta, \sin\theta)\|^2 = \cos^2\theta + \sin^2\theta = 1$$

$$\|(\cos\theta, \sin\theta)\| = 1$$

Q rotates vectors by the angle θ



Ex 2 Permutations

Permutation matrices are a reordering of the columns of I.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Ex 3 Reflection

Let $\vec{u}$ be any unit vector.

$$Q = I - 2\vec{u}\vec{u}^T.$$

$$\begin{aligned} Q^TQ &= (I - 2\vec{u}\vec{u}^T)^T Q \\ &= (I - 2\vec{u}\vec{u}^T) Q \\ &= Q - 2\vec{u}\vec{u}^T Q \end{aligned}$$

$$= I - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T\left(I - 2\vec{u}\vec{u}^T\right)$$

$$= I - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4\vec{u}\underbrace{\vec{u}^T\vec{u}}_{=1}\vec{u}^T$$

$$= I - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T$$

$$= I$$

Let $\vec{u} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

$$Q = I - 2\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

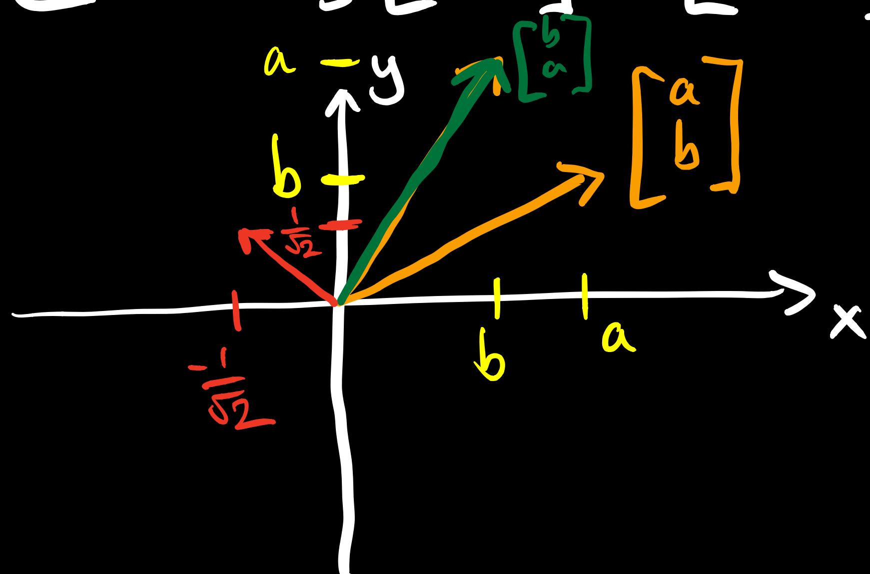
$$= I - 2\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



Projections Using Orthonormal Bases

Recall the projection matrix:

$$P = A(A^TA)^{-1}A^T$$

Now, replacing A with Q, we have

$$\begin{aligned} P &= Q(Q^TQ)^{-1}Q^T \\ &= QIQ^T \\ &= QQ^T \end{aligned}$$

<u>Note</u>: When Q is square, $P = I$, so the projection $\vec{b}$ onto the whole space is $\vec{b}$ and

$$\begin{aligned} P\vec{b} &= QQ^T\vec{b} \\ &= \vec{q}_1(\vec{q}_1^{\,T}\vec{b}) + \cdots + \vec{q}_n(\vec{q}_n^{\,T}\vec{b}) \end{aligned}$$

We see that $\vec{b}$ is the sum

of its components along the $\vec{q}_i$'s.

Recall the least squares solution was $$\hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}$$

Now, $$\hat{\vec{x}} = (Q^T Q)^{-1} Q^T \vec{b}$$
$$= Q^T \vec{b}$$

The Gram-Schmidt Process

We have seen that orthogonality is nice. Given a basis or some collection of vectors, we can make them orthonormal using the Gram-Schmidt Process.

Gram-Schmidt Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a collection of linearly independent vectors.

1. Orthogonalize

$$\vec{w}_1 = \vec{v}_1$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_2}{\|\vec{w}_2\|^2}\vec{w}_2 - \frac{\vec{v}_3 \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1$$

$$\vdots$$

$$\vec{w}_n = \vec{v}_n - \sum_{i=1}^{n-1} \frac{\vec{v}_n \cdot \vec{w}_i}{\|\vec{w}_i\|^2}\vec{w}_i$$

2. Normalize:

$$\vec{q}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|}, \ \vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}, \ldots, \vec{q}_n = \frac{\vec{w}_n}{\|\vec{w}_n\|}$$

$\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$ form an orthonormal set of vectors that span the same space as $\vec{v}_1, \ldots, \vec{v}_n$.

The Factorization $A=QR$

We can write $A=QR$ where Q has orthonormal columns and R is upper triangular.

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \vec{q}_1^T\vec{a}_1 & \vec{q}_1^T\vec{a}_2 & \cdots & \vec{q}_1^T\vec{a}_n \\ & \vec{q}_2^T\vec{a}_2 & & \vdots \\ & & \ddots & \\ & & & \vec{q}_n^T\vec{a}_n \end{bmatrix}$$

$$A = QR$$

We can use Gram-Schmidt on the columns of A to find Q.

$$A = QR$$
$$Q^TA = Q^TQR$$
$$= R$$