Chapter 5 Determinants

5.1 The Properties of Determinants

The determinant of a matrix is a number which captures a lot of important information about the matrix. One of the most important properties is that the determinant is zero when a matrix has no inverse.

The Properties of the Determinant

Let's look at some properties of the determinant. We will examine the 2x2 case, but the properties hold for the nxn case.

The determinant of a 2x2 matrix is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = \underline{ad} - \underline{bc}$$

We also write

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

1. The determinant of the identity matrix is 1.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

2. The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$= -(bc - ad)$$

$$= - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

Then $\det P = 1$ when P exchanges an even number of rows

$\det(P) = -1$ when P exchanges an odd number of rows

3. The determinant is a linear function of each row.

$$\begin{vmatrix} ta + a' & tb + b' \\ c & d \end{vmatrix} = (ta + a')d - (tb + b')c$$

$$= tad + a'd - tbc - b'c$$

$$= t(ad - bc) + (a'd - b'c)$$

$$= t\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

<u>Note</u> $\det(tA) \neq t\det(A)$. tA multiplies each row of A by t, so

$$\det(tA) = t^n \det(A)$$

4. If two rows of A are equal, then $\det(A) = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$$

Since $\det(A) = 0$, A is not invertible. We see that the columns of A are linearly dependent.

5. Subtracting a multiple of one row from another row leaves $\det(A)$ unchanged.

$$\begin{vmatrix} a & b \\ c - la & d - \ell b \end{vmatrix} = a(d - \ell b) - b(c - \ell a)$$

$$= ad \cancel{- \ell ab} - bc \cancel{+ \ell bc}$$

$$= ad - bc$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Remark We see that the steps of elimination from A to U does not change the determinant if no permutations needed. Then $\det(A) = \det(U)$.

Each time a row swap is performed the determinant changes sign, so

$$\det(A) = \pm \det(U)$$

6. A matrix with a row of zeros has $\det(A) = 0$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = a \cdot 0 - b \cdot 0 = 0$$

7. If A is triangular, then $\det(A) = a_{11} a_{22} \cdots a_{nn}$

= product of the diagonal entries

$$\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac - b \cdot 0 \qquad \begin{vmatrix} a & 0 \\ b & c \end{vmatrix} = ac - b \cdot 0$$

$$= ac \qquad\qquad = ac$$

Remark We can compute $\det(A)$ by row reducing

A to U since rule 5 says $\det(A) = \pm\det(U)$.

8. If A is singular, then $\det(A) = 0$. If A is invertible, then $\det(A) \neq 0$.

   Recall that if A is singular then U has a zero row. Then by rule 6, $\det(U) = 0$, and by rule 5, $\det(A) = \pm\det(U) = 0$.

9. $\det(AB) = \det(A)\det(B)$

   This is tedious to show, but you will prove the 2×2 case in PS11.

   Note If A is invertible, then

$$
\begin{aligned}
AA^{-1} &= I \\
\det(AA^{-1}) &= \det(I) \\
\det(A)\det(A^{-1}) &= 1 \\
\det(A^{-1}) &= \frac{1}{\det(A)}
\end{aligned}
$$

10. The transpose of A has the same determinant as A. $\det(A) = \det(A^T)$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
$$= \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

<u>Remark</u> Every rule for the rows can apply to the columns since $|A| = |A^T|$.

Now, let's see how to compute determinants.

5.2 Permutations and Cofactors

We will examine 3 ways to compute $\det(A)$.

The Pivot Formula

We can row reduce A to find $A = LU$. L has 1's on its diagonal and let $d_1, \dots, d_n$ be the diagonal entries of U. Then

$$\begin{aligned} \det(A) &= \det(LU) \\ &= \det(L)\det(U) \\ &= 1 \cdot d_1 \cdot d_2 \cdots d_n \end{aligned}$$

If a row swap was required, then $PA = LU$, so $$\det(P)\det(A) = \det(L)\det(U)$$

and so $$\det(A) = \pm\det(U).$$

<u>Ex</u>

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

P performs 1 row exchange, so $\det(P) = -1$

$$\det(P)\det(A) = 1 \cdot 4 \cdot 6$$
$$-\det(A) = 24$$
$$\det(A) = -24$$

The Big Formula for Determinants

The Big Formula has $n!$ terms. The number of terms grows quickly, $10! = 3{,}628{,}800$.

$$\text{Big Formula} = \det(A)$$

$$= \sum \det(P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

= sum over $n!$ column permutations
$P = (\alpha, \beta, \ldots, \omega)$.

Let's look at the 3x3 case. $3! = 6$ orderings of columns.

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{21} a_{12} a_{33}$$
$$+ a_{13} a_{21} a_{32} + a_{31} a_{12} a_{23}$$
$$- a_{31} a_{22} a_{13}$$

Determinant by Cofactors

Cofactor Formula: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

$$C_{ij} = (-1)^{i+j} \det M_{ij}$$

where M_{ij} is an $(n-1)\times(n-1)$ submatrix of A that has row i and column j removed.

Remark 1. This is the expansion along row i, we may also expand along columns.

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

2. Use the column or row with the most zeros!

3. Signs of $(-1)^{i+j} = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

Ex $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix}$

$$\det(A) = 1\cdot\det\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix}_{①} - 0 + 1\cdot\det\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}_{②}$$

$$-1\cdot\det\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}_{③}$$

①
$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 2 \end{vmatrix} = 1\cdot\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 4-3-(0-3)$$

$$= 4$$

②
$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 3 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= 2(0-3) - 1(2-0) + 1(3-0)$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= -5$$

③
$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 0 - 2\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$= -1(1-1) - 2(2-1)$

$= -2$

$\det(A) = 4 + {-5} - {-2}$

$= 1$