Chapter 6: Eigenvalues and Eigenvectors

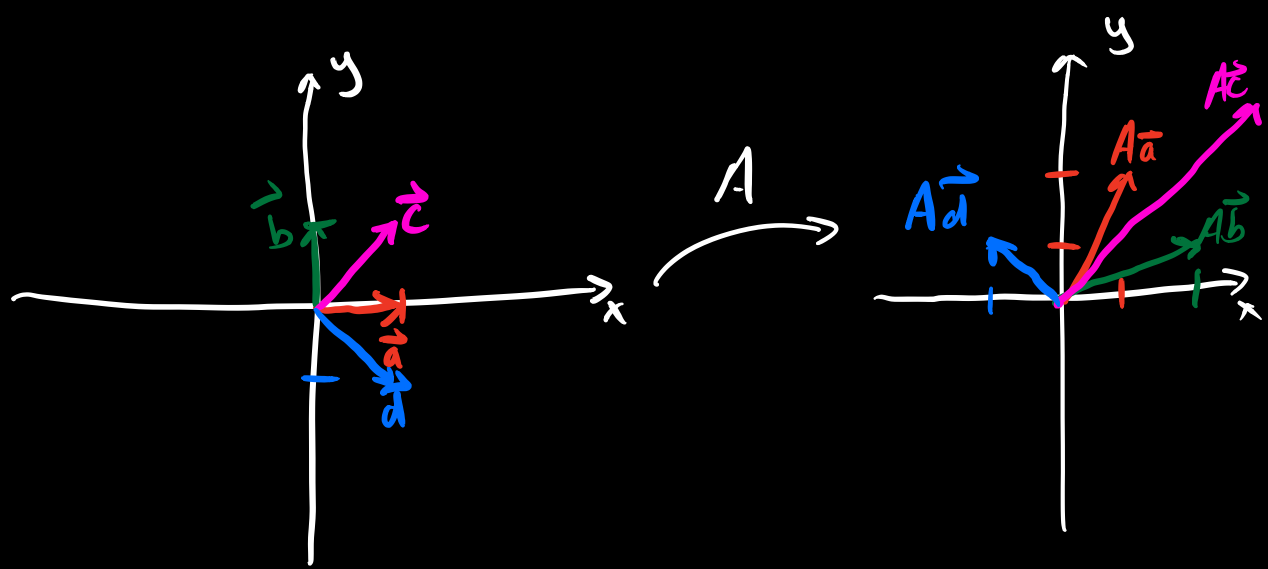
6.1 Introduction to Eigenvalues

Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Let's see how A acts on some vectors.

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A\vec{c} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad A\vec{d} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Notice $A\vec{c}$ points in the same direction as $\vec{c}$ and $A\vec{d}$ lies on the same line as $\vec{d}$. That is

$$A\vec{c} = 3\vec{c}$$
$$A\vec{d} = -\vec{d}$$

$\vec{c}$ and $\vec{d}$ are eigenvectors of A, and $3, -1$ are eigenvalues of A.

Now, let's see how to find eigenvectors and eigenvalues.

The Equation for the Eigenvalues

Definition An eigenvalue of an $n \times n$ matrix A is a complex number λ such that there is a nonzero vector $\vec{v}$ with $A\vec{v} = \lambda\vec{v}$. $\vec{v}$ is called an eigenvector, and $\text{Span}(\vec{v})$ is the eigenspace associated with λ.

Remark 1. We see that

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$
so $\vec{v} \in N(A - \lambda I)$.

2. Since $\vec{v} \neq \vec{0}$, then $A - \lambda I$ is singular, so $\det(A - \lambda I) = 0$. This is the eigenvalue equation. We find the roots of the polynomial $\det(A - \lambda I)$.

3. λ is an eigenvalue of A if and only if $A - \lambda I$ is singular.

4. To find the eigenvector $\vec{v}$, solve
$$(A - \lambda I)\vec{v} = \vec{0} \quad \text{or} \quad A\vec{v} = \lambda\vec{v}$$

<u>Ex</u> Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -3 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Compute the determinant of $A-\lambda I$

$$\det(A-\lambda I) = \det \begin{array}{cc} \begin{array}{c} + \\ - \\ + \\ \end{array} & \begin{array}{c} \begin{bmatrix} 2-\lambda & 0 & -1 \\ -3 & -1-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{bmatrix} \\ \begin{array}{ccc} & - & + \end{array} \end{array} \end{array}$$

$$= (1-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ -3 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(-1-\lambda)$$

2. Find the roots of the polynomial.

$$(1-\lambda)(2-\lambda)(-1-\lambda) = 0$$

$$\left.\begin{array}{l} \lambda = 1 \\ \lambda = 2 \\ \lambda = -1 \end{array}\right\} \text{These are the eigenvalues of } A.$$

3. For each eigenvalue λ, solve $(A-\lambda I)\vec{v} = \vec{0}$ to find $\vec{v}$.

$\underline{\lambda = 1}$

$$(A-I)\vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ -3 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \vec{0}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ -3 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} r_2 + 3r_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} -\frac{1}{2} r_2$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \vec{0}$$

$$v_1 - v_3 = 0 \implies v_1 = v_3$$

$$v_2 = 0$$

$$\vec{v} = \begin{bmatrix} v_1 \\ 0 \\ v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvector associated with eigenvalue $\lambda = 1$ is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

<u>Note</u> If $\vec{v}$ is an eigenvector, so is $c\vec{v}$.

$$A\vec{v} = \lambda\vec{v}$$

$$A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$$

$$A(c\vec{v}) = \lambda(c\vec{v})$$

So $c\vec{v}$ is an eigenvector of A.

<u>$\lambda = 2$</u>

$$(A-2I)\vec{v} = \vec{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ -3 & -3 & 3 \\ 0 & 0 & -1 \end{bmatrix}\vec{v} = \vec{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ -3 & -3 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{array}{l} r_1 - r_3 \\ r_2 + 3r_3 \\ \\ \end{array} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -3 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} -\frac{1}{3}r_1 \\ -r_2 \\ \\ \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \vec{0}$$

$$v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$$

$$v_3 = 0$$

$$\vec{v} = \begin{bmatrix} v_1 \\ -v_1 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is the eigenvector associated with eigenvalue

$\lambda = 2.$

$\underline{\lambda = -1}$

$$(A + I)\vec{v} = \vec{0}$$

$$\begin{bmatrix} 3 & 0 & -1 \\ -3 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \vec{v} = \vec{0}$$

Notice $\vec{v} = v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the eigenvector associated with eigenvalue $\lambda = -1$.

<u>Summary</u> Our eigenvalue-eigenvector pairs are

$$1, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad -1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let's look at

$$\begin{array}{c} + \\ - \\ + \end{array} \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ \boxed{1} & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$$

$$= 1$$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible, so the columns are linearly independent. We have 3 linearly independent vectors in $\mathbb{R}^3$, so $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ form a basis of $\mathbb{R}^3$. This is called the eigenbasis.

Determinant and Trace

The determinant of A is the product of n eigenvalues.

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n .$$

Ex

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -3 & -1 & 3 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix}$$

$$= -2$$

A had eigenvalues $1, 2, -1$, and

$$1 \cdot 2 \cdot -1 = -2$$

So we see that indeed

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 .$$

Defn The trace of A is the sum of the diagonal entries

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

The trace of A is the sum of the n eigenvalues of A.

$$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Ex

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -3 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$tr(A) = 2 - 1 + 1$$
$$= 2$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + -1$$
$$= 2$$

So we see that indeed

$$tr(A) = \lambda_1 + \lambda_2 + \lambda_3$$

Remark The diagonal of a triangular matrix reveals its eigenvalues! The eigenvalues <u>are</u> the diagonal entries.

<u>Ex</u> $U = \begin{bmatrix} \lambda_1 & u_{12} & \cdots & u_{1n} \\ & \lambda_2 & \ddots & \vdots \\ & & \ddots & u_{n-1,n} \\ 0 & & & \lambda_n \end{bmatrix}$

$$\det(U-\lambda I) = \begin{vmatrix} \lambda_1-\lambda & u_{12} & \cdots & u_{1n} \\ & \lambda_2-\lambda & \ddots & \vdots \\ & & \ddots & u_{n-1,n} \\ 0 & & & \lambda_n-\lambda \end{vmatrix}$$

$$= (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$

Then $\det(U-\lambda I)=0$, so

$\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$.

<u>Some Facts About Eigenvalues</u>

1. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k, and λ^{-1} is an eigenvalue of A^{-1}.
2. If $\lambda=0$ is an eigenvalue of A, then

A is singular since
$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n = 0$$

3. Projections, reflections, and rotations have the eigenvalues $1, 0, -1, i, -i$. That is, $|\lambda| = 1$ or 0.

6.2 Diagonalizing a Matrix

We have seen factorizations of A. Now, we decompose A using its eigenvalues and eigenvectors.

Diagonalization Suppose the $n \times n$ matrix A has n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Put them into the columns of an eigenvector matrix X. Then $X^{-1}AX$ is the eigenvalue matrix Λ:

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Why is $AX = X\Lambda$?

$$AX = A\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$= X\Lambda$$

Thus, $AX = X\Lambda$ and we may write

$$X^{-1}AX = \Lambda \quad \text{or} \quad A = X\Lambda X^{-1}$$

<u>Note</u> We need n independent eigenvectors to diagonalize A.

What about diagonalizing A^k?

$$A^k = (X \Lambda X^{-1})^k$$
$$= X \Lambda X^{-1} X \Lambda X^{-1} \cdots X \Lambda X^{-1}$$
$$= X \Lambda I \Lambda I \cdots I \Lambda X^{-1}$$
$$= X \Lambda^k X^{-1}$$