Warm-up Diagonalize $A = \begin{bmatrix} 4 & -2 & -3 \\ 3 & -1 & -3 \\ 2 & -2 & -1 \end{bmatrix}$, $A\vec{v} = \vec{v}$.

Since $A\vec{v} = \vec{v}$, $\lambda = 1$ is an eigenvalue of A.

1. Find eigenvalues of A.

$$
\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \begin{vmatrix} 4-\lambda & -2 & -3 \\ 3 & -1-\lambda & -3 \\ 2 & -2 & -1-\lambda \end{vmatrix} \\
&= (4-\lambda)\begin{vmatrix} -1-\lambda & -3 \\ -2 & -1-\lambda \end{vmatrix} + 2\begin{vmatrix} 3 & -3 \\ 2 & -1-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & -1-\lambda \\ 2 & -2 \end{vmatrix} \\
&= (4-\lambda)\left[(-1-\lambda)^2 - 6\right] + 2\left[3(-1-\lambda) + 6\right] \\
&\qquad - 3\left[-6 - 2(-1-\lambda)\right] \\
&= (4-\lambda)\left[\lambda^2 + 2\lambda - 5\right] + 2\left[3 - 3\lambda\right] \\
&\qquad - 3\left[2\lambda - 4\right] \\
&= -\lambda^3 + 2\lambda^2 + 13\lambda - 20 + 6(1-\lambda) - 6(\lambda - 2) \\
&= -\lambda^3 + 2\lambda^2 + \lambda - 2
\end{aligned}
$$

Since $\lambda = 1$ is an eigenvalue of A, so $(\lambda - 1)$ divides $-\lambda^3 + 2\lambda^2 + \lambda - 2$.

$$
\begin{array}{r|l}
 & -\lambda^2 + \lambda + 2 \\
\hline
\lambda - 1 & -\lambda^3 + 2\lambda^2 + \lambda - 2 \\
 & \underline{-\lambda^3 + \lambda^2} \\
 & \qquad \lambda^2 + \lambda \\
 & \qquad \underline{\lambda^2 - \lambda} \\
 & \qquad\qquad 2\lambda - 2 \\
 & \qquad\qquad \underline{2\lambda - 2} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

$$
\begin{aligned}
-\lambda^3 + 2\lambda^2 + \lambda - 2 &= (\lambda - 1)(-\lambda^2 + \lambda + 2) \\
&= (\lambda - 1)(\lambda - 2)(\lambda + 1) \cdot -1 \\
&= 0
\end{aligned}
$$

$\lambda = 1$, $\lambda = 2$, $\lambda = -1$ are the eigenvalues of A.

2. Find the eigenvectors

$\underline{\lambda = 1}$ $(A - I)\vec{v} = \vec{0}$

$$
\begin{bmatrix} 3 & -2 & -3 \\ 3 & -2 & -3 \\ 2 & -2 & -2 \end{bmatrix} \vec{v} = \vec{0}, \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
$$

$\underline{\lambda=2}$ $(A-2I)\vec{v}=\vec{0}$

$$\begin{bmatrix} 2 & -2 & -3 \\ 3 & -3 & -3 \\ 2 & -2 & -3 \end{bmatrix}\vec{v}=\vec{0}, \quad \vec{v}=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda=-1}$ $(A+I)\vec{v}=\vec{0}$

$$\begin{bmatrix} 5 & -2 & -3 \\ 3 & 0 & -3 \\ 2 & -2 & 0 \end{bmatrix}\vec{v}=\vec{0}, \quad \vec{v}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3. Write down the diagonalization of A.

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}}_{\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Remark 1. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct, then $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent. A matrix with no repeated eigenvalues can be diagonalized.

2. The eigenvectors in X come in the same order as the eigenvalues in Λ

3. Some matrices do not have enough eigenvectors to be diagonalized.

Similar Matrices: Same Eigenvalues

In the factorization $X \Lambda X^{-1}$, what happens when we change the eigenvector matrix X? This is the idea behind similar matrices.

Defn Two matrices A and C are called similar if there exists an invertible matrix B such that

$$A = B^{-1} C B$$

$A = X \Lambda X^{-1}$
So A and Λ are similar.

Prop Similar matrices have the same eigenvalues.

Proof Suppose A and C are similar matrices. Then there exists a matrix B so that

$$A = BCB^{-1}.$$

Let $\lambda, \vec{v}$ be an eigenvalue and eigenvector of A. Then

$$A\vec{v} = \lambda\vec{v}.$$

Now,

$$BCB^{-1}\vec{v} = \lambda\vec{v}$$

$$C\underbrace{B^{-1}\vec{v}} = \lambda\underbrace{B^{-1}\vec{v}}$$

and so λ is an eigenvalue of C with eigenvector $B^{-1}\vec{v}$.

Nondiagonalizable Matrices

Not all matrices are diagonalizable. The issue becomes not having enough eigenvectors.

When an eigenvalue is repeated, there is a need for more than just 1 eigenvector.

There are two ways to count the multiplicity of λ.

1. Geometric Multiplicity (GM) - Count the independent eigenvectors for λ

$* \; GM = \dim(N(A-\lambda I))$

2. Algebraic Multiplicity (AM) - Count the repititions of λ as a solution to $\det(A-\lambda I)=0$.

Notice : $GM \leq AM$

Ex $A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}$

$$\det(A-\lambda I) = (6-\lambda)(4-\lambda)+1$$
$$= \lambda^2 - 10\lambda + 25$$

$$\lambda^2 - 10\lambda + 25 = 0$$
$$(\lambda-5)^2 = 0$$
$\lambda = 5$, Algebraic multiplicity : 2

$$(A-5I)\vec{v} = \vec{0}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Row reduce $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \to$ rank 1, so nullity $= 2-1 = 1$.

Geometric multiplicity : 1

6.4 Symmetric Matrices

At first we considered general matrices that did not have any underlying properties. We have seen a few classes of matrices (triangular, orthogonal), and now we consider symmetric matrices.

1. A symmetric matrix has only real eigenvalues.
2. The eigenvectors can be chosen to be orthonormal.

Theorem (Spectral Theorem) Every symmetric matrix has the factorization $S = Q\Lambda Q^T$ with real eigenvalues in Λ and orthonormal eigenvectors in the columns of Q.

Symmetric diagonalization: $S = Q\Lambda Q^{-1}$
$= Q\Lambda Q^T$

Ex $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$
$$(1-\lambda)^2 - 4 = 0$$
$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda - 3)(\lambda + 1) = 0$$
$$\lambda = 3 \quad \lambda = -1.$$

$\lambda = 3$

$$(A - 3I)\vec{v} = \vec{0}$$
$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\vec{v} = \vec{0}$$
$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \|\vec{v}\| = \sqrt{2}$$

$\lambda = -1$

$$(A + I)\vec{v} = \vec{0}$$
$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\vec{v} = \vec{0}$$
$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \|\vec{v}\| = \sqrt{2}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Ex 2 Show that the eigenvalues of a real 2x2 symmetric matrix are real.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + ad - bc \end{aligned}$$

ax^2+bx+c

"$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$"

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$= \frac{(a+d) \pm \sqrt{a^2+d^2-2ad+4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(a-d)^2+4bc}}{2}$$

A is symmetric so $b=c$. Then

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2+4b^2}}{2}$$

$(a-d)^2 + 4b^2 \geq 0$, so $\lambda \in \mathbb{R}$

Symmetric Matrices as sums of rank 1 matrices

$$S = Q \Lambda Q^{-1}$$
$$= Q \Lambda Q^T$$
$$= \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} - & \vec{q}_1^T & - \\ - & \vec{q}_2^T & - \\ & \vdots & \\ - & \vec{q}_n^T & - \end{bmatrix}$$
$$= \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 \vec{q}_1^T \\ \lambda_2 \vec{q}_2^T \\ \vdots \\ \lambda_n \vec{q}_n^T \end{bmatrix}$$
$$= \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \cdots + \lambda_n \vec{q}_n \vec{q}_n^T$$

6.5 Positive Definite Matrices

A symmetric matrix S whose eigenvalues are all strictly positive is called positive definite.

A test for positive definite-ness is

$$\vec{x}^T A \vec{x} > 0$$

for all $\vec{x} \neq \vec{0}$.