1.3 Matrices

Definition A $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$ is an $m \times n$ array of numbers with m rows and n columns. The entries of A are $a_{ij} \in \mathbb{R}$ and $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Remarks 1. Capital letters will typically denote matrices

2. (a_{ij}) is another way to represent A.

Ex 1. $A \in \mathbb{R}^{3 \times 3}$, $a_{ij} = i + j$ ←

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

when $m=n$, A is called a square matrix.

2. $B \in \mathbb{R}^{1\times 3}$, $a_{ij} = 1$

a_{13}
a_{12}
a_{13}

$$B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

We see that B is a row vector

<u>Question</u> How do matrices and vectors interact?

<u>Answer</u> A matrix A acts on a vector $\vec{x}$ through matrix-vector multiplication which produces a new vector $A\vec{x}$.

Definition Let $A \in \mathbb{R}^{m\times n}$ and $\vec{x} \in \mathbb{R}^n$.
The matrix A acts on vector $\vec{x}$ through matrix vector multiplication. The resulting vector $\vec{v} \in \mathbb{R}^m$ has entries

$$v_i = \sum_{j=1}^{n} a_{ij} x_j$$

Remarks 1. The matrix $A \in \mathbb{R}^{m\times n}$ acts on $\vec{x} \in \mathbb{R}^l$ for $l = n$.

2. The definition is a bit clunky, so how else can we think about performing matrix vector multiplication?

3. The entries of the resulting vector $\vec{v} = A\vec{x}$, are found by taking the dot product of $\vec{x}$ and the rows of A.

Ex Let

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1) + 1\cdot 0 + 2\cdot 1 \\ (-1)\cdot 0 + 1\cdot 0 + 1\cdot 0 \\ (-1)(1) + 0\cdot 0 + 0\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

A change in perspective: We can also think of matrix-vector multiplication as a new vector being produced by taking a linear combination of the columns of A where the scalar multiplied to the j^{th} column is the j^{th} entry of $\vec{x}$.

Ex Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \cdot 1 + x_2 \cdot 0 + x_3 \cdot 0 \\ x_1 \cdot 0 + x_2 \cdot 1 + x_3 \cdot 0 \\ x_1 \cdot 1 + x_2 \cdot 1 + x_3 \cdot 1 \end{bmatrix}$$

Ex Let $A \in \mathbb{R}^{m \times n}$, $m = n$.

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \quad \} \text{ diagonal} \\ -1 & \text{if } i-1 = j, \quad i = 2, \dots, n \\ 0 & \text{else} \end{cases}$$

and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$i = 2$
$j = 2 - 1 = 1$
a_{21}

$$A = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

Let $\vec{b} = A\vec{x}$. Then

$$b_i = \sum_{j=1}^{n} a_{ij} x_j$$

$$= a_{i1} x_1 + a_{i2} x_2 + \cdots\cdots$$

$$+ a_{i\,i-1} x_{i-1} + a_{ii} x_i + a_{i\,i+1} x_{i+1}$$

$$+ \cdots + a_{in} x_n$$

$$= a_{i\,i-1} x_{i-1} + a_{ii} x_i$$

$$= -1 \cdot x_{i-1} + x_i$$

$$= x_i - x_{i-1}$$

Then

$$\vec{b} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_2 - x_3 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

Note: Be careful with the first entry

Here, A is called a difference matrix because $\vec{b}$ contains the differences of the input $\vec{x}$.

Linear Equations

Consider the generic equation

$$A\vec{x} = \vec{b}.$$

Until now, we have considered A and $\vec{x}$ to be known. What if we consider the inverse problem? That is, what if we know A and $\vec{b}$ and want to find $\vec{x}$?

Ex Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

We want to find $\vec{x}$ so that $A\vec{x} = \vec{b}$.

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We get 3 equations for 3 unknowns

$$x_1 + x_3 = b_1$$
$$x_2 + x_3 = b_2$$
$$x_3 = b_3$$

$$A\vec{x} = \vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

We see that

$$\boxed{\begin{aligned} x_1 &= b_1 - b_3 \\ x_2 &= b_2 - b_3 \\ x_3 &= b_3. \end{aligned}} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{A^{-1}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A\vec{x} = \vec{0}$

If $\vec{b} = \vec{0}$, notice that means that $\vec{x} = \vec{0}$. This means that A is invertible and we can recover $\vec{x}$ from $\vec{b}$ and $\vec{x} = \underbrace{A^{-1}}_{\text{inverse of } A}\vec{b}$

Note this is not true for all matrices.

Independence and Dependence and the Inverse Matrix

<u>Definition</u> Let $\vec{v}_i$, $i=1,2,\ldots,K$ be a collection of vectors. The collection $\vec{v}_i$ is linearly independent if

$$\sum_{i=1}^{K} c_i \vec{v}_i = \vec{0}$$

if and only if $c_i = 0$ for $i=1,\ldots,K$.
If the $\vec{v}_i$ are not linearly independent, we say that they are dependent.

How does this relate to the inverse matrix?

If the matrix A has independent columns, then A is an invertible matrix, and $A\vec{x}=\vec{0}$ has only one solution.

If the matrix A has dependent columns, then A is called a singular matrix and $A\vec{x}=0$ has infinitely many or no solution.

$$\sum_{i=1}^{K} c_i \vec{v}_i = \vec{0}$$

$$\sum_{i=1}^{K} c_i \vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_K\vec{v}_K$$

$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_K \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \end{bmatrix}$$

Ch. 2 Solving Linear Equations

2.1 Vectors and Linear Equations

Consider the linear system of m equations and n unknowns.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The matrix form of the equations is

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

A is called the coefficient matrix.

<u>Recall</u> We have two perspectives to think about $A\vec{x}$

1. Dot product of $\vec{x}$ and the rows of A
2. A linear combination of the columns of A with coefficients x_j.

The Geometric Perspective

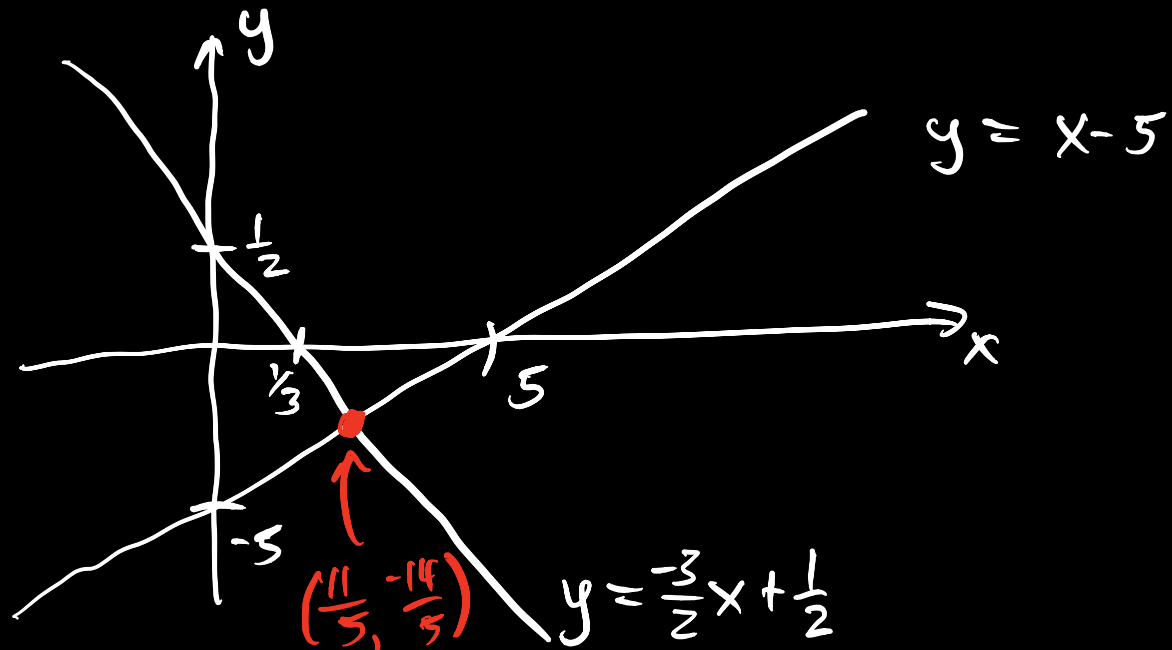
<u>2D</u>

Consider
$$3x+2y=1$$
$$x-y=5$$

$$\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

We have two equations of a line. The

Solution (x,y) is the point of intersection of the lines.



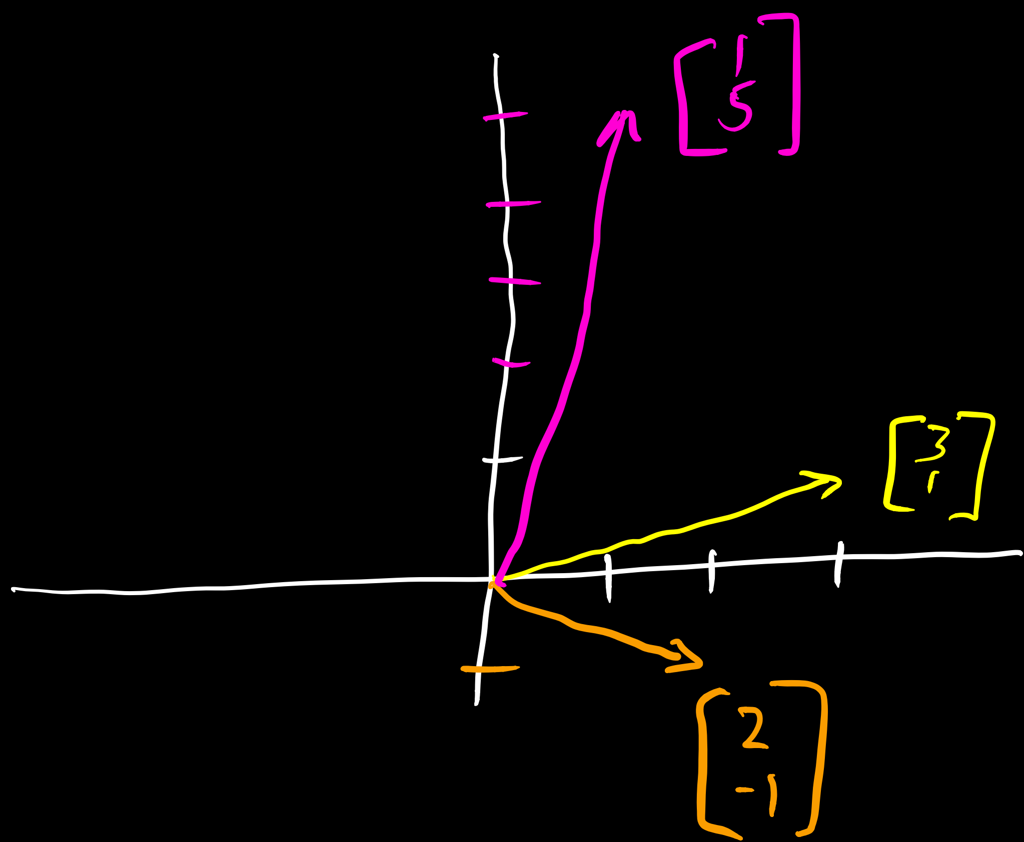
$$3x+2y=1 \qquad\qquad 3x+2x-10=1$$
$$y=x-5 \quad \Rightarrow \quad 5x=11$$
$$x=\frac{11}{5}$$

$$y=\frac{11}{5}-5=\frac{-14}{5}$$

This is the row picture

Let's now consider the column picture.

$$x\begin{bmatrix}3\\1\end{bmatrix}+y\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$$



We want to find x and y to produce the desired vector.

The column picture wants to find a linear combination of the columns of A to produce a vector $\vec{b}$.

3D How does the 2D interpretation translate to 3D?

Row picture: Each row is an equation of a plane. The solution is the point of intersection of three planes.

Column picture: We want a linear combination of the columns to produce a desired vector.

A Special Matrix

Definition $I_n \in \mathbb{R}^{n \times n}$ where

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

is the identity matrix.

Remark 1. You can think of the identity matrix as the same role that 1 plays in multiplication. That is,

$$I\vec{x} = \vec{x}.$$

Ex 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \cdot 1 + 0 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2. $I = \delta_{ij} = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & \text{otherwise} \end{array} \right]$

Kronecker delta

$$I \vec{x} = \vec{v}$$

$$v_i = \sum_{j=1}^{n} \delta_{ij} x_j$$

$$= \delta_{ii} x_i$$

$$= x_i$$

$v_i = x_i$ for all i

so $\vec{x} = \vec{v}$.