2.2 The idea of Elimination

Last time We introduced systems of linear equations. We had m equations for n unknowns, and we had the matrix form of the system

$$A\vec{x} = \vec{b}$$

where $A \in \mathbb{R}^{m \times n}$, $\vec{x} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$.

We want to develop a way to solve these systems.

Consider the following system

$$2x + y = 6 \quad (1)$$
$$x + y = 3 \quad (2)$$

Let's eliminate x by adding (1) to $-2(2)$. Then we get

$$\begin{aligned} \boxed{2}x + y &= 6 \\ \boxed{-}y &= 0 \end{aligned} \qquad \begin{bmatrix} \boxed{2} & 1 \\ 0 & \boxed{-1} \end{bmatrix}$$

Notice the structure of the system.
It is an upper triangular system.
Now, we can back substitute
to find x.

$$y = 0$$
$$x = \frac{1}{2}(6 - y) = 3$$

For a general system, we want to

1. Form an upper triangular system
2. Back substitute

How do we get to an upper triangular system?

We use pivots! A pivot is the

first nonzero entry in a row used to eliminate the entries below it.

The pivots are on the diagonal of the triangle after elimination.

<u>Notation</u> The matrix U will typically denote an upper triangular matrix.

<u>Elimination from A to U</u>

<u>Procedure</u>: Use a_{ii} as a pivot. The multiplier

$$l_{ki} = \frac{a_{ki}}{a_{ii}} \begin{array}{l} \leftarrow \text{what are want to eliminate} \\ \leftarrow \text{pivot} \end{array}$$

for $k = i+1, \ldots, m$.

Then use l_{ki} to zero out a_{ki}.

Ex

$$\boxed{2}x + 2y + 3z = 1 \quad (1)$$

$$2\cdot\tfrac{1}{2}-1=0 \Rightarrow \boxed{1}x + 3y + 4z = 2 \quad (2)$$

$$\boxed{-2}x + y + z = 3 \quad (3)$$

Pivot : 2

$$l_{21} = \frac{1}{2} \qquad l_{31} = \frac{-2}{2}$$

$$l_{21}(1) - (2) \Rightarrow -2y - \frac{5}{2}z = \frac{-3}{2}$$

$$l_{31}(1) - (3) \Rightarrow -3y - 4z = -4$$

$$2x + 2y + 3z = 1 \quad (1)$$

$$\boxed{-2}y - \frac{5}{2}z = \frac{-3}{2} \quad (2)$$

$$-3y - 4z = -4 \quad (3)$$

Pivot : -2

$$l_{32} = \frac{-3}{-2}$$

$$l_{32}(2) - (3) \Rightarrow \frac{1}{4}z = \frac{7}{4}$$

We have

$$\begin{aligned} 2x + 2y + 3z &= 1 \\ -2y - \tfrac{5}{2}z &= -\tfrac{3}{2} \\ \tfrac{1}{4}z &= \tfrac{7}{4} \end{aligned}$$

Now, we back substitute

$$z = 7$$
$$y = -\tfrac{1}{2}\left(-\tfrac{3}{2} + \tfrac{5}{2}z\right) = -8$$
$$x = \tfrac{1}{2}(1 - 2y - 3z) = -2$$

Solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ 7 \end{bmatrix}$

When does elimination breakdown?

1. What if the pivot is zero?
   We can exchange rows!

$\underline{Ex}$ $\boxed{0}x + 3y = 6$
$2x + y = 1$

Exchanging rows, we have

$$2x + y = 1$$
$$0x + 3y = 6$$

2. There could be no solution.

$\underline{Ex}$ $\boxed{2}x + y = 2 \quad (1)$
$4x + 2y = 7 \quad (2)$

$$l_{21} = \frac{4}{2}$$

$$l_{21}(1) - (2) \Rightarrow 0y = -3$$

$$2x + y = 2$$
$$0y = -3$$

No solution!

Row picture: Parallel lines have no

point of intersection!

3. There could be infinitely many solutions!

<u>Ex</u> $\boxed{2}x + y = 2 \quad (1)$

$4x + 2y = 4 \quad (2)$

$\ell_{21} = \frac{4}{2}$

$\ell_{21}(1) - (2) \Rightarrow 0y = 0$

$$2x + y = 2$$
$$0y = 0$$

Here, y is called "free". Once y is chosen, we have

$$x = -\frac{1}{2}(2 - y)$$

- Elimination succeeds when we have n pivots, but we may have to exchange rows/equations.

Next, we would like to use matrices to solve linear systems.

First, lets examine how matrices interact.

2.4 Rules for Matrix Operations

Let A be an $m \times n$ matrix and B an $l \times k$ matrix.

Matrix addition: We can add A and B if $m = l$ and $n = k$

Let $C = A + B$. Then

$$c_{ij} = a_{ij} + b_{ij}$$

- addition is entry-wise.

Matrix Multiplication

We can multiply A and B if $n = \ell$.
That is, A has as many columns as B has rows.

Definition Let A be $m \times n$ and B $n \times \ell$.
Then the product AB produces a matrix C with m rows and ℓ columns. C has entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{k\ell}$$

Remark 1. $c_{ij} = $ (row i of A) $\cdot$ (column j of B)

Ex $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+1(3) & 2(2)+1(4) \\ 1(1)+2(3) & 2(1)+2(4) \\ 0(1)+1(3) & 2(0)+1(4) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 \\ 7 & 10 \\ 3 & 4 \end{bmatrix}$$

Note that BA is not defined!

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \quad !??!$$

<u>Ex</u> An "extreme" example

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= 1 + 4 + 9 + 16$$

$$= 30$$

A column times a row is an "outer" product

A row times a column is an "inner" product.

Three More Perspectives for Matrix Multiplication

2. Matrix A times every column of $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_\ell \end{bmatrix}$

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & Ab_\ell \end{bmatrix}$$

3. Every row of A times B

$$A = \begin{bmatrix} - \vec{a}_1 - \\ - \vec{a}_2 - \\ \vdots \\ - \vec{a}_m - \end{bmatrix}$$

$$AB = \begin{bmatrix} \vec{a}_1 B \\ \vec{a}_2 B \\ \vdots \\ \vec{a}_m B \end{bmatrix}$$

4. Multiply columns of A to rows of B and sum.

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \quad B = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_n \end{bmatrix}$$

$$AB = \vec{a}_1 \vec{b}_1 + \vec{a}_2 \vec{b}_2 + \cdots + \vec{a}_n \vec{b}_n$$

Laws for Matrix Operations

Addition

$A+B = B+A$ (commutative)

$c(A+B) = cA + cB$ (distributive)

$A+(B+C) = (A+B)+C$ (associative)

Multiplication

$A(B+C) = AB+AC$ (distributive)

$(A+B)C = AC+BC$

$A(BC) = (AB)C$ (associative)

Note $AB \neq BA$

Matrix multiplication is not commutative!

Suppose A is $n \times n$. Then

$$A^p = AAA \cdots A \quad (p \text{ factors})$$

and

$$(A^p)(A^q) = A^{p+q}$$

$$(A^p)^q = A^{pq}$$

Note 1. A^{-1} is the inverse of A, not I/A

2. $A^0 = I$.

2.3 Elimination Using Matrices

Now we know how to work with matrices, so we want to solve a linear using matrices!

We want to obtain a triangular

matrix, and to do this we need to "create zeros". The elimination matrix E_{ij} will be our key tool.

<u>Definition</u> The elimination matrix E_{ij} is the matrix with entries 1 on the diagonal, $-l_{ij}$ in row i column j and 0 everywhere else. Here, l_{ij} is the multiplier associated with pivot a_{ii}.

<u>Ex</u>

$$\begin{aligned} 2x + y &= 6 \\ x + y &= 3 \end{aligned} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$l_{21} = \frac{1}{2}$$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

We see that E_{21} creates the desired zero!

<u>Remark</u> E_{ij} created a zero in the matrix $E_{ij}A$

We may also need to exchange rows, so how do we do that?

<u>The Permutation Matrix</u>

The permutation matrix P_{ij} is a permutation of the identity matrix. That is, the i^{th} row and j^{th} row of the identity matrix are swapped.

Ex 3×3 $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$$P_{23}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

Ex $0x+3y=6$

$2x+y=1$

$$\begin{bmatrix}0&3\\2&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}6\\1\end{bmatrix}$$

$$P_{12}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

$$P_{12}\begin{bmatrix}0&3\\2&1\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&3\\2&1\end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

Now, let's use the elimination matrix and Permutation matrix to solve a linear system.

$$A\vec{x} = \vec{b}$$

$$E_{ij} A\vec{x} = E_{ij}\vec{b}$$

$$P_{ij} E_{ij} A\vec{x} = P_{ij} E_{ij}\vec{b}$$

$$E_{kl} P_{ij} E_{ij} A\vec{x} = E_{kl} P_{ij} E_{ij}\vec{b}$$

$$\vdots$$

$$E_{pq} \cdots P_{ij} \cdots E_{ij} A\vec{x} = E_{pq} \cdots P_{ij} \cdots E_{ij}\vec{b}$$

There is a sequence of matrices C_i so that

$$C_1 C_2 \cdots C_q A$$

is upper triangular.