Some loose ends:

A matrix / system of equations is singular if there are infinitely many solutions or no solution.

A nonsingular matrix / system has a full set of pivots and exactly one solution.

## The Augmented Matrix

Consider $A\vec{x} = \vec{b}$. To solve this system, we can use the augmented matrix

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$$

Elimination acts on whole rows of

this matrix.

Ex
$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

pivot: 2

$l_{31} = \frac{1}{2}$

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ \boxed{1} & 1 & 1 & 1 \end{array}\right] r_3 - \tfrac{1}{2} r_1$$

pivot: 1

$l_{32} = \frac{\frac{1}{2}}{1}$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & \boxed{\frac{1}{2}} & \frac{3}{2} & \frac{1}{2} \end{array}\right] r_3 - \tfrac{1}{2} r_2$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right]$$

Thus, we have

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Using back substitution, we find

$$x_3 = 0$$
$$x_2 = 1 - 2x_3 = 1$$
$$x_1 = \frac{1}{2}(1 - x_2 + x_3) = 0$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

This approach to solving systems is much more common, but we will see elimination matrices expose a powerful decomposition of A.

## 2.5 Inverse Matrices

We have been looking at systems of linear equations of the form $A\vec{x}=\vec{b}$. We have seen that under certain conditions, the inverse of $A$, denoted $A^{-1}$, exists and we may write

$$\vec{x}=A^{-1}\vec{b}.$$

How can we find $A^{-1}$ and when does $A^{-1}$ exist?

<u>Definition</u> The matrix $A$ is invertible if there exists a matrix $A^{-1}$ so that

$$A^{-1}A=AA^{-1}=I.$$

When does the inverse matrix exist?

1. The inverse exists if and only if elimination produces $n$ pivots.

2. If $A$ is invertible, then there is only one solution to $A\vec{x} = \vec{b}$

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

3. If there is a nonzero $\vec{x}$, so that $A\vec{x} = \vec{0}$, then $A$ is not invertible.

4. If $A$ is invertible, then $A\vec{x}=\vec{0}$ can only have the solution $\vec{x}=\vec{0}$.

<u>Remark</u> 1. If $A$ is invertible, then $A^{-1}$ is unique. Suppose $BA=I$ and $AC=I$. Then

$$B(AC)=(BA)C$$
$$BI=IC$$
$$B=C$$

2. The inverse of a $2\times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3. Suppose $A$ is a diagonal matrix. Then

if $A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix}$

then

$$A^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1/d_n \end{bmatrix}$$

<u>Note</u> We must have that $d_i \neq 0$ for $i = 1, 2, \ldots, n$.

## The Inverse of a product $AB$

If $A$ and $B$ are invertible, then so is $AB$, and

$$(AB)^{-1} = B^{-1} A^{-1}.$$

**Ex** What is the inverse of the product $A_1 A_2 A_3 A_4 \cdots A_n$?

Let $B_1 = A_2 A_3 \cdots A_n$. Then

$$\begin{aligned}(A_1 A_2 A_3 \cdots A_n)^{-1} &= (A_1 B_1)^{-1} \\ &= \underbrace{B_1^{-1}}_{?} A_1^{-1}\end{aligned}$$

Let $B_2 = A_3 A_4 \cdots A_n$. Then

$$\begin{aligned}B_1^{-1} &= (A_2 A_3 \cdots A_n)^{-1} \\ &= (A_2 B_2)^{-1} \\ &= B_2^{-1} A_2^{-1}\end{aligned}$$

So

$$(A_1 A_2 \cdots A_n)^{-1} = B_2^{-1} A_2^{-1} A_1^{-1}$$

$$\vdots$$

$$(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \cdots A_2^{-1} A_1^{-1}$$

# Calculating $A^{-1}$ by Gauss-Jordan

## Elimination

We want to compute $A^{-1}$. We know

$$AA^{-1} = I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ be the columns of $A^{-1}$ and $\vec{e}_i = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$ $\quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Then

$$A \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix}$$

so $A\vec{x}_i = \vec{e}_i$ and we have $n$ systems to solve. The Gauss-Jordan method computes

$A^{-1}$ by solving all $n$ equations together. We use the augmented matrix

$$\left[\begin{array}{c:c} A & I \end{array}\right]$$

and we want

$$\left[\begin{array}{c:c} I & A^{-1} \end{array}\right]$$

$$A A^{-1} = I$$

$$(E_{ij} E_{kl} \cdots E_{pq} A) A^{-1} = E_{ij} E_{kl} \cdots E_{pq} I$$

$$I A^{-1} = E_{ij} E_{kl} \cdots E_{pq}$$

In Gauss-Jordan elimination, the entries above the pivots are also zeroed out. We go all the way to reduced echelon form

$$\begin{bmatrix} I & B \end{bmatrix}.$$

<u>Ex</u> Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Augmented Matrix

$$\begin{array}{c} \text{pivot: } 2 \\ l_{31} = \frac{1}{2} \\ \phantom{x} \end{array} \left[\begin{array}{ccc|ccc} \boxed{2} & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ \boxed{1} & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \phantom{x} \\ \phantom{x} \\ r_3 - \frac{1}{2} r_1 \end{array}$$

$$\begin{array}{c} \text{pivot: } 1 \\ l_{12} = \frac{1}{1} \\ l_{32} = \frac{-\frac{1}{2}}{1} \end{array} \left[\begin{array}{ccc|ccc} 2 & \boxed{1} & 2 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & \boxed{-\frac{1}{2}} & 0 & -\frac{1}{2} & 0 & 1 \end{array}\right] \begin{array}{l} r_1 - 1 \cdot r_2 \\ \phantom{x} \\ r_3 + \frac{1}{2} r_2 \end{array}$$

$$\begin{array}{c} \text{pivot: } \frac{1}{2} \\ l_{13} = \frac{1}{\frac{1}{2}} \\ l_{23} = \frac{1}{\frac{1}{2}} \end{array} \left[\begin{array}{ccc|ccc} 2 & 0 & \boxed{1} & 1 & -1 & 0 \\ 0 & 1 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{\frac{1}{2}} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \begin{array}{l} r_1 - 2 r_3 \\ r_2 - 2 r_3 \\ \phantom{x} \end{array}$$

$$\begin{array}{cc} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & -2 & -2 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] & \begin{array}{l} r_1/2 \\ \\ r_3 \cdot 2 \end{array} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

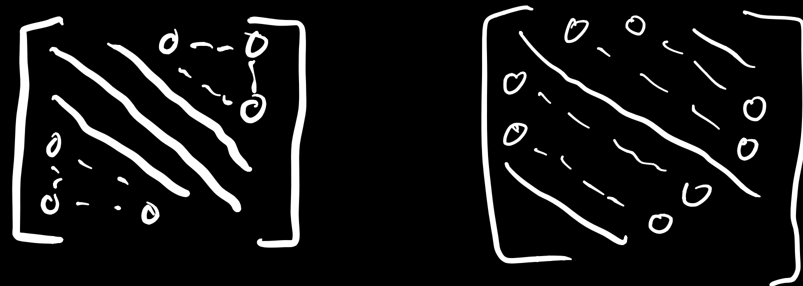
Thus,

$$A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

Some observations about inverse matrices:

1. If $A$ is symmetric ($a_{ij} = a_{ji}$), then $A^{-1}$ is also symmetric.

2. If $A$ is tridiagonal, then $A^{-1}$ is dense with no zeros.

$A =$ 

3. The product of the pivots is the determinant of $A$.

## 2.6 Elimination = Factorization: $A = LU$

Given a matrix $A$, we previously used elimination matrices to arrive at an upper triangular matrix $U$.

$$E_{ij} E_{kl} \cdots E_{pq} A = U$$

$$A = (E_{ij} E_{kl} \cdots E_{pq})^{-1} U$$

$$= E_{pq}^{-1} \cdots E_{kl}^{-1} E_{ij}^{-1} U$$

$= LU$

where $L$ is an upper triangular matrix. We see that we can write $A$ in a factored form $A = LU$.

<u>Note</u> There could be a permutation required, but we assume that no row exchanges were needed.

$\boxed{A = LU}$

- $U$ is an upper triangular matrix with the pivots on the diagonal.
- $L$ is a lower triangular matrix with 1's on the diagonal and the multipliers $l_{ij}$ below the diagonal.

__Ex__ $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} \boxed{2} & 1 & 0 \\ \boxed{1} & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\overset{E_{31}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}} \begin{bmatrix} \boxed{2} & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ \boxed{1} & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \boxed{\frac{3}{2}} & 1 \\ 0 & \boxed{\frac{1}{2}} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

So $(E_{32}\,E_{31}\,E_{21})A = U$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = U$$

$$A = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1} U$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}^{-1} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} U$$

$$= L U$$

we see that the nonzero entries of $L$ are $1$ on the diagonal and $l_{ij}$ below the diagonal. Also, the pivots are on the diagonal of $U$.